Zbiory przeliczalne

Def. X jest przeliczalny $\equiv X = \emptyset \vee$

$$(\exists f)\left(f : \mathbb{N} \xrightarrow{na} X\right)$$

Uwaga: $f : \mathbb{N} \xrightarrow{na} X$

$$X = f[\mathbb{N}] = \{f(n) : n \in \mathbb{N}\}$$

$$a_n = f(n)$$

$$X = \{a_n : n \in \mathbb{N}\}$$

P. $\mathbb{N}$ są przelicz, bo $Id_{\mathbb{N}} : \mathbb{N} \xrightarrow{na} \mathbb{N}$

Tw. X jest przeliczalny $\equiv (\exists n \in \mathbb{N})(|X| = n) \vee |X| = \aleph_0$

D-d. ⇐ ① $|X| = n$ $\quad X \sim \{0, .., n-1\}, \; n \geq 1$

$$f : \{0, .., n-1\} \xrightarrow[1-1]{na} X$$

$$f^*(k) = \begin{cases} f(k) & : k \leq n-1 \\ f(n-1) & : k \geq n \end{cases}$$

$$X = \{a_0, ..., a_{n-1}\}$$

$$\begin{array}{lccccccc} X: & a_0, & a_1, & ..., & a_{n-1}, & a_{n-1}, & a_{n-1}, & ...... \\ & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & \\ f^* & 0 & 1 & & n-1 & n & n+1 & \end{array}$$

② $|X| = \aleph_0$

$|X| = |\mathbb{N}|$

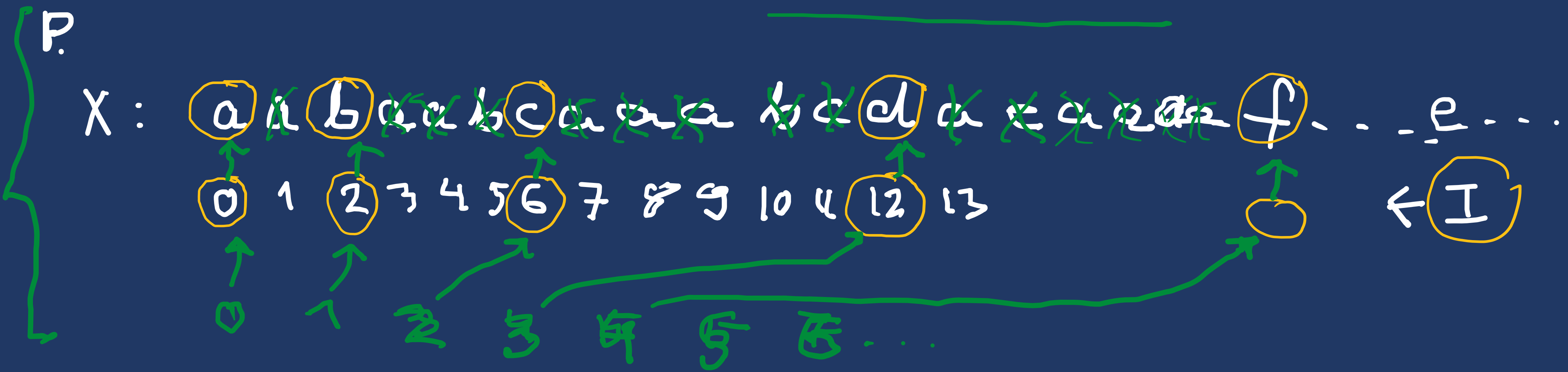
$$f : \mathbb{N} \xrightarrow{na} X$$

☒

$\Rightarrow \quad X = \emptyset \vee (\exists f : \mathbb{N} \xrightarrow{na} X) \Rightarrow (\exists n)(|X| = n) \vee |X| = |\mathbb{N}|$

Zał. że $f : \mathbb{N} \xrightarrow{na} X$ i $\neg(\exists n)(|X| = n)$.

P.



Niech $I = \{k \in \mathbb{N} : (\forall l < k)(f(l) \neq f(k))\}$

$(= \{k \in \mathbb{N} : f(k) \notin \{f(0), .., f(k-1)\}\})$

Uwaga : $a \in X$; $\{k : f(k) = a\} \neq \emptyset \subseteq \mathbb{N}$

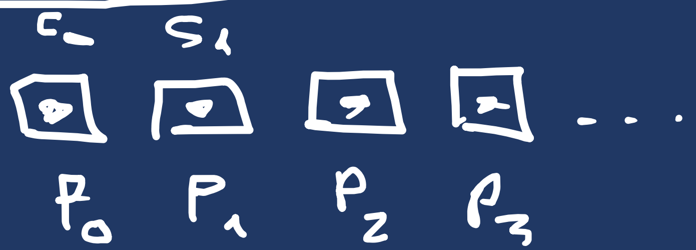
1) $I \subseteq \mathbb{N}$

2) I – jest niesk.

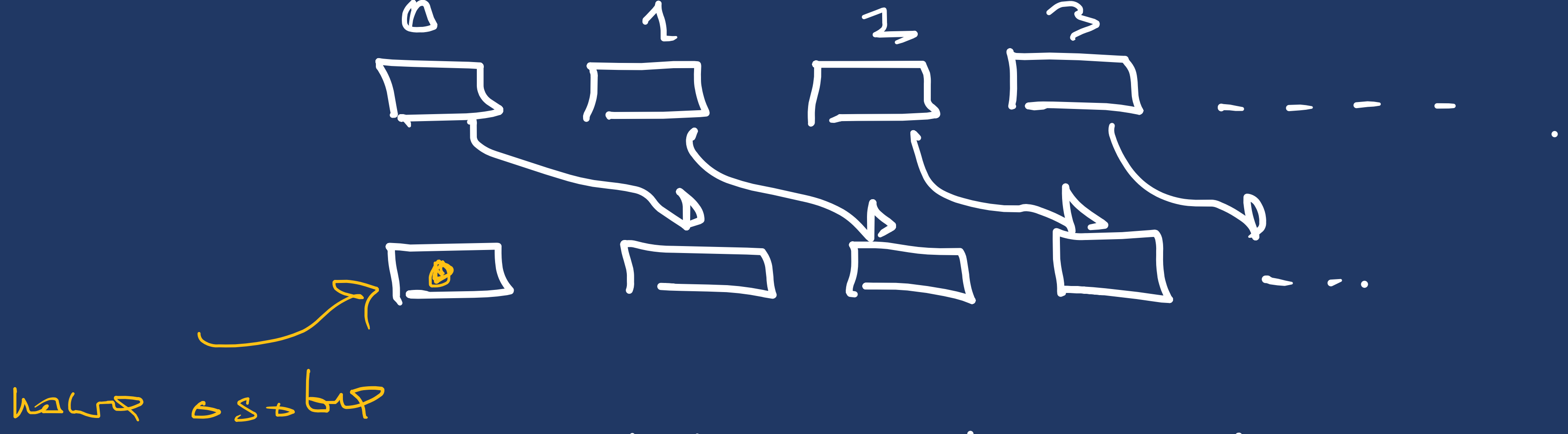
$I: \; n_0 < n_1 < n_2 < \dots$ $\qquad I = \{n_i\}_{i \in \mathbb{N}}$

Def. $\; h(i) = f(n_i) \qquad i \in \mathbb{N}$

Ⓟ hotel: $\{p_0, p_1, \dots\}$



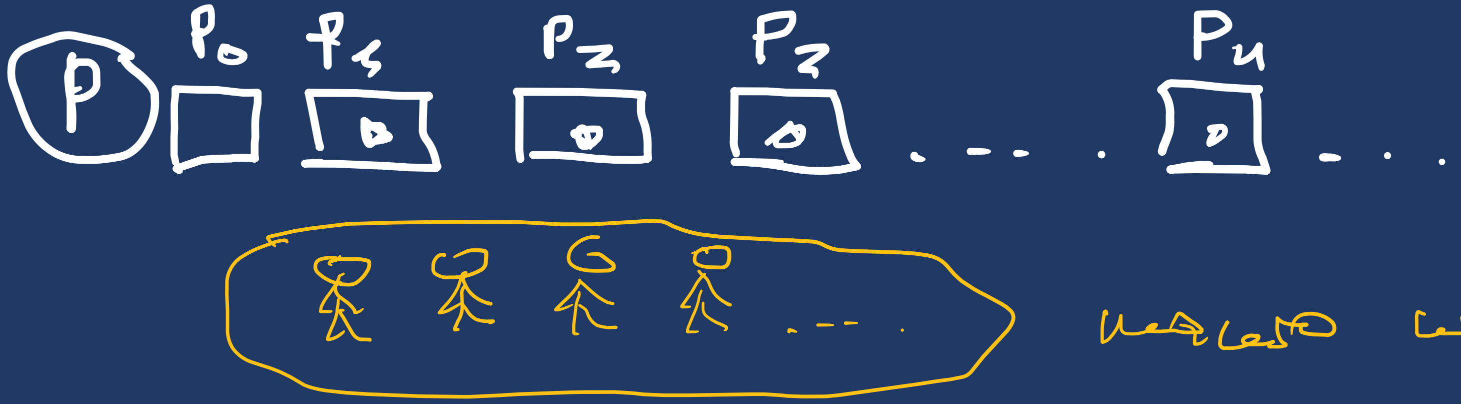

← nowa osoba



$$|\mathbb{N}^+| = |\mathbb{N}|$$

$$f(n) = n+1$$

$$f: \mathbb{N} \xrightarrow[\text{na}]{1-1} \mathbb{N}^+$$



[illegible] [illegible] nowych $\aleph_0$

$$|2\cdot\mathbb{N}| = |\mathbb{N}|$$

$$f(n) = 2\cdot n$$

$$f: \mathbb{N} \xrightarrow[\text{na}]{1-1} \{2n : n\in\mathbb{N}\}$$

$|2\cdot N| = |2\cdot N+1| = |N|$

Geom. aksj. (Euklides)

Aksj: "część jest mniejsza od całości"

Kontr. aksj. wymagają dopisu,

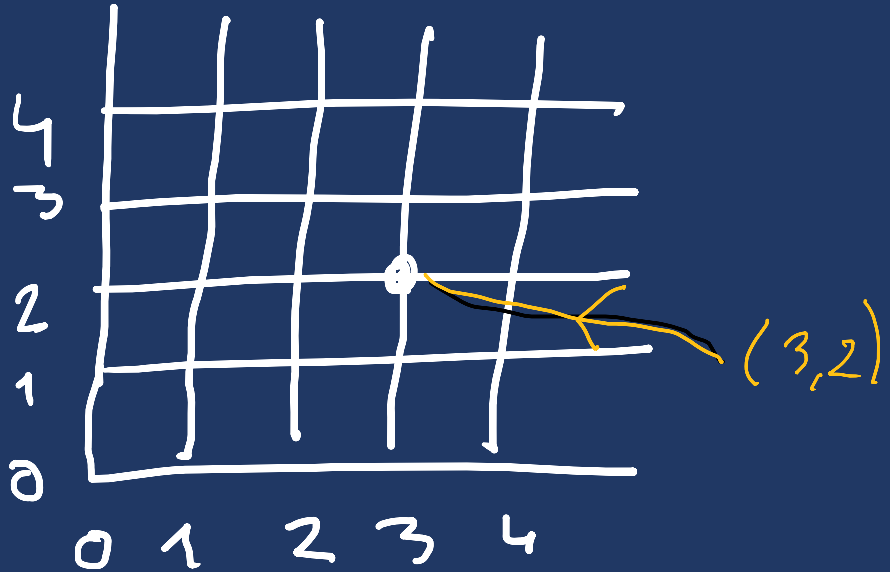
$|N\setminus\{0\}| = |N|$


W0
W1
W2
?
P0
P1
P2

Tw. $\boxed{|\mathbb{N}\times\mathbb{N}| = \aleph_0}$

D-d. $f: \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}$

$$f((n,k)) = 2^n(2\cdot k+1) - 1.$$



1) f jest 1-1 :

$$f((n_1,k_1)) = f((n_2,k_2))$$

$$2^{n_1}(2k_1+1)\underline{\underline{-1}} = 2^{n_2}(2k_2+1)\underline{\underline{-1}}$$

$$2^{n_1}(2k_1+1) = 2^{n_2}(2k_2+1)$$

$$\downarrow$$

$$2^{n_1} \mid 2^{n_2},\ 2^{n_2} \mid 2^{n_1} \longrightarrow n_1 = n_2$$

$$2k_1+1 = 2k_2+1 \longrightarrow k_1 = k_2$$

$$(n_1,k_1) = (n_2,k_2)$$

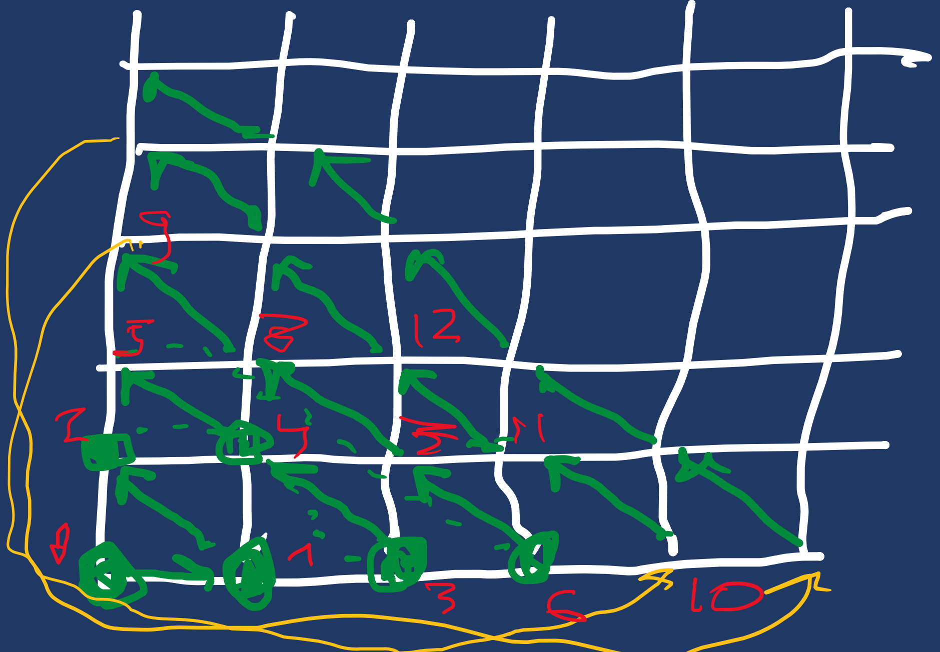
2) f jest "na" ; $\qquad f((n,k)) = 2^n(2k+1)-1$

weźmy $a \in \mathbb{N}$

$a+1 = 2^n(2k+1)$ dla pewnych $n,k \in \mathbb{N}$

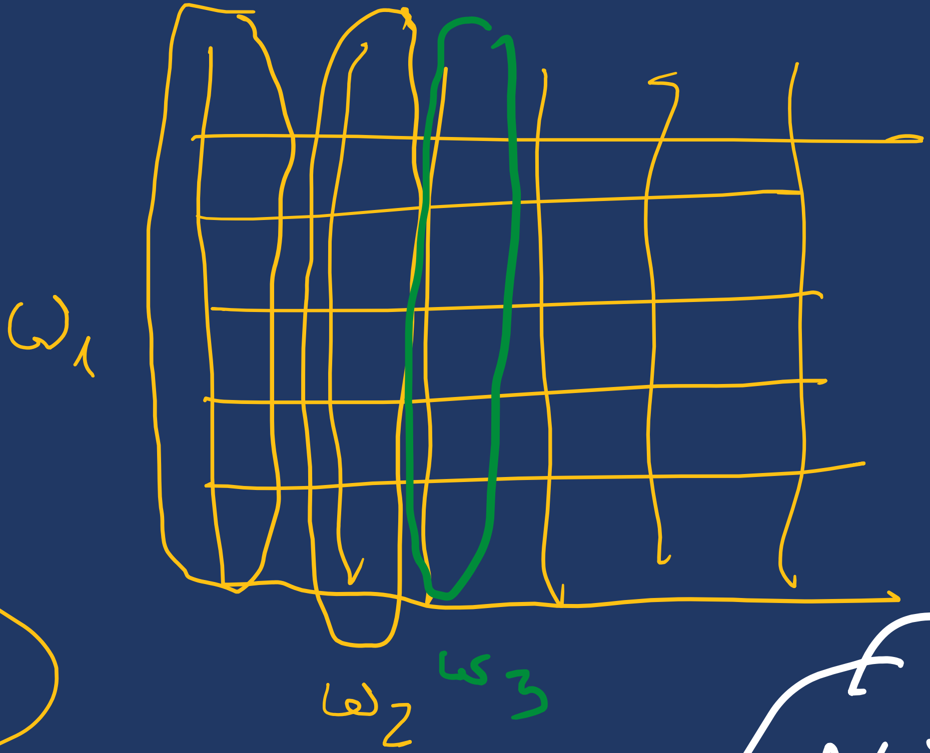
wtedy $f((n,k)) = 2^n(2k+1) - 1 = (a+1) - 1 = a$

$$f : \mathbb{N} \times \mathbb{N} \xrightarrow[\text{na}]{1-1} \mathbb{N}$$



$\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

Hotele



$$\varphi : \mathbb{N} \xrightarrow[\text{na}]{1-1} \mathbb{N} \times \mathbb{N}$$

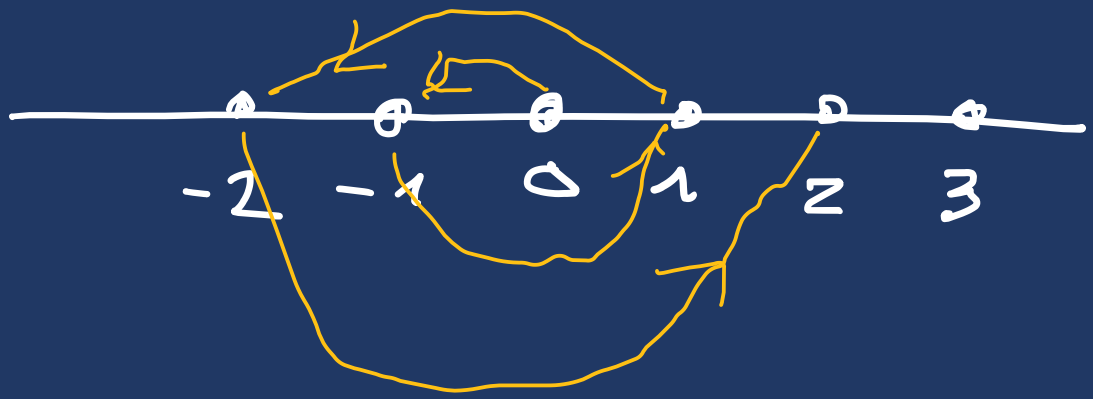
$$\varphi(n) = (\varphi_1(n), \varphi_2(n))$$

$$f(n) = W_{\varphi_1(n), \varphi_2(n)}$$

$$W_i = (\omega_{i,0}, \omega_{i,1}, \omega_{i,2} \ldots), i \in \mathbb{N}$$

Ⓟ $|\mathbb{Z}| = \aleph_0$

$$f(n) = \begin{cases} k : & n = 2k \\ -k-1 : & n = 2k+1 \end{cases}$$



$f(0) = 0$

$f(1) = -1$

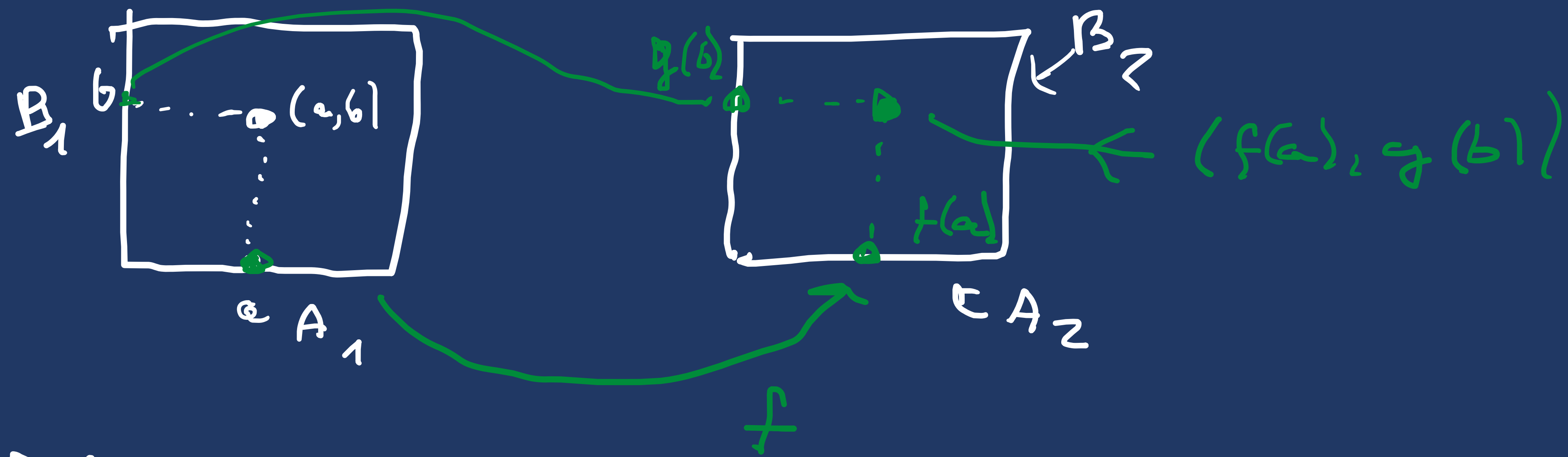
$f(2) = 1$

$f(3) = -2$

$\vdots$

FAKT $\left.\begin{array}{l}|A_1| = |A_2| \\ |B_1| = |B_2|\end{array}\right\} \Rightarrow |A_1 \times B_1| = |A_2 \times B_2|$

D-d. $f: A_1 \xrightarrow[\text{na}]{1-1} A_2$, $g: B_1 \xrightarrow[\text{na}]{1-1} B_2$



Definiujemy: $\varphi((a,b)) = (f(a), g(b))$

$\varphi: A_1 \times B_1 \longrightarrow A_2 \times B_2$

- φ jest 1-1:

$$\varphi((a,b)) = \varphi((c,d))$$
$$\downarrow$$
$$(f(a), g(b)) = (f(c), g(d))$$
$$\downarrow$$
$$f(a) = f(c) \wedge g(b) = g(d) \to a = c \wedge b = d$$
$$\to (a,b) = (c,d).$$

- φ jest "na" $A_2 \times B_2$.

Weźmy $(c,d) \in A_2 \times B_2$.

- $c \in A_2$: jest $a \in A_1$ t. że $f(a) = c$
- $d \in B_2$: jest $b \in B_1$ t. że $g(b) = d$.

Wtedy

$$\varphi((a,b)) = (f(a), g(b)) = (c,d) \quad \blacksquare$$

czyli $\varphi : A_1 \times B_1 \xrightarrow[\text{na}]{1-1} A_2 \times B_2$

czyli $|A_1 \times B_1| = |A_2 \times B_2|$.

$\blacksquare$

Ⓟ $|\mathbb{Z} \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = \aleph_0$.

$\mathbb{Z} \sim \mathbb{N}$

$\mathbb{N} \sim \mathbb{N}$

Ⓟ $\varphi : \mathbb{Z} \times \mathbb{N} \xrightarrow{\text{na}} \mathbb{Q} : (k, n) \to \frac{k}{n+1}$

$\mathbb{Q}$ - przelicz.

$\mathbb{Q}$ nie jest skończ.

$$\boxed{|\mathbb{Q}| = \aleph_0}$$

$$\begin{tikzcd}
\mathbb{Z} \times \mathbb{N} \arrow[r, "\varphi", "\text{na}"'] & \mathbb{Q} \\
\mathbb{N} \arrow[u, "\eta", "\text{na}"'] \arrow[ur, "\text{„na''}", "\varphi \circ \eta"'] &
\end{tikzcd}$$

Tw. Zał. że $(A_n)_{n\in\mathbb{N}}$ są zbiorami przeliczalnymi, wtedy

$$\bigcup_n A_n$$

jest zbiorem przeliczalnym.

? (Ac) ?

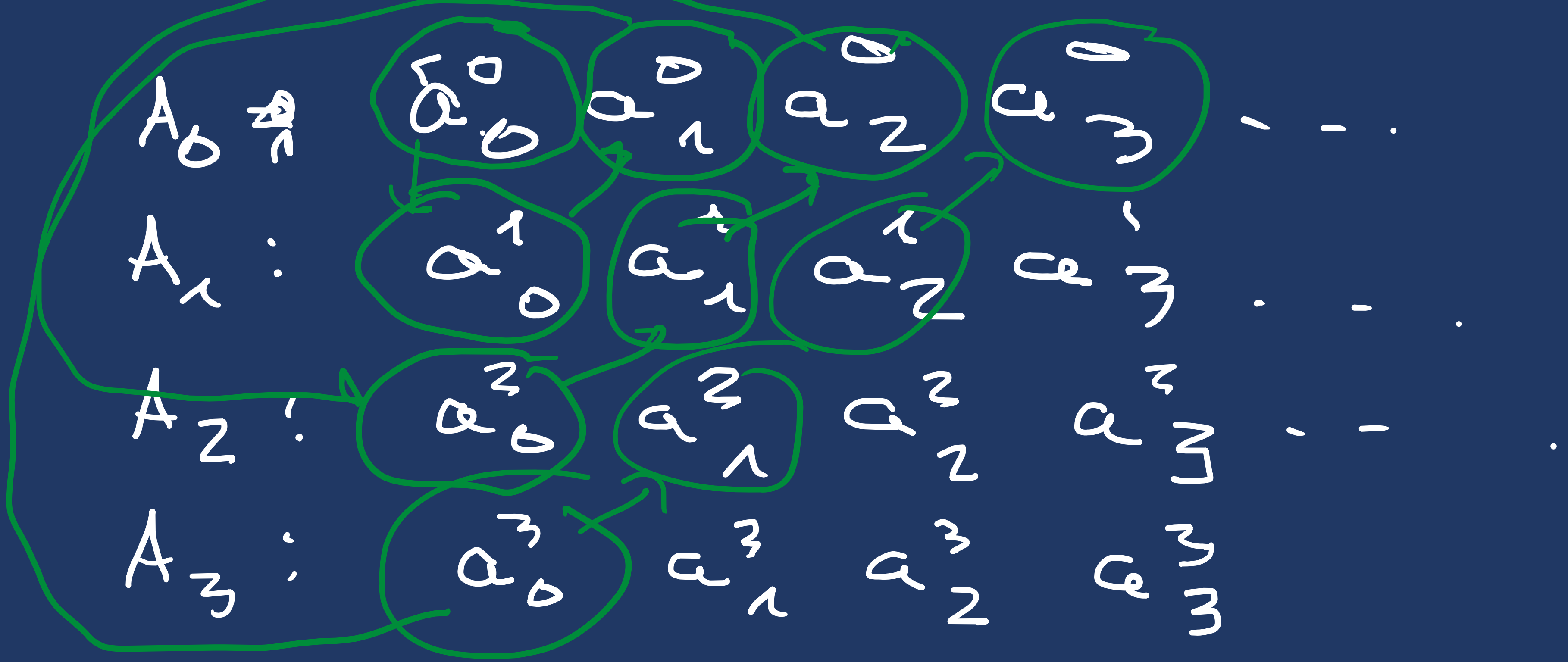
D-d. Redukcja: możemy zał. że $(\forall n)(A_n \neq \emptyset)$.

- dla każdego n ustalamy $\varphi_n : \mathbb{N} \xrightarrow{na} A_n$.
- bierzemy $\psi = (\psi_1, \psi_2) : \mathbb{N} \xrightarrow{na} \mathbb{N} \times \mathbb{N}$

DEFINIUJEMY:

$$F(n) = \varphi_{\psi_1(n)}(\psi_2(n)).$$

TO JEST TO!!!



$$\begin{array}{lllll}
A_0 : & a^0_0 & a^0_1 & a^0_2 & a^0_3 \; \ldots \\
A_1 : & a^1_0 & a^1_1 & a^1_2 & a^1_3 \; \ldots \\
A_2 : & a^2_0 & a^2_1 & a^2_2 & a^2_3 \; \ldots \\
A_3 : & a^3_0 & a^3_1 & a^3_2 & a^3_3
\end{array}$$