Arytmetyka kardynalna

Przykłady: $0, 1, 2, \ldots, \aleph_0, \mathfrak{c}, \ldots$

$\aleph_0 = |\mathbb{N}|, \quad \mathfrak{c} = |\mathbb{R}|$

Def. Zał. że $|X| = \kappa$, $|Y| = \lambda$

1) $\kappa + \lambda = |(X \times \{0\}) \cup (Y \times \{1\})|$

2) $\kappa \cdot \lambda = |X \times Y|$

3) $\kappa^{\lambda} = |X^{Y}|$

Uwagi: 1) $(X \times \{0\}) \cap (Y \times \{1\}) = \emptyset$

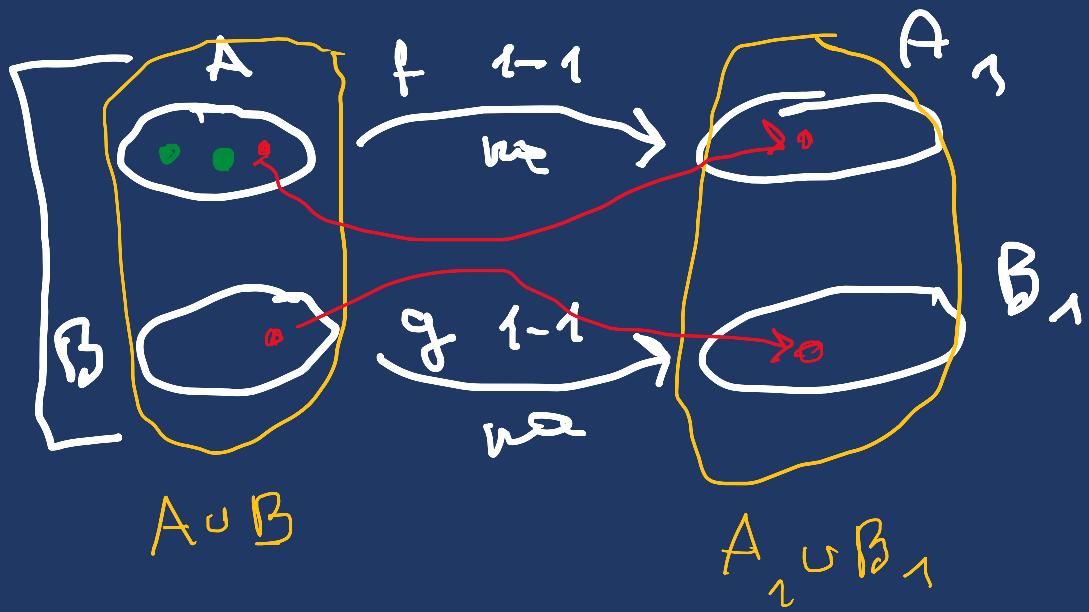
2) $\varphi : X \longrightarrow X \times \{a\}$; $\varphi(x) = (x, a)$

wtedy $\varphi : X \xrightarrow[\text{na}]{1-1} X \times \{a\}$; więc $X \sim X \times \{a\}$

$|X| = |X \times \{a\}|$

Lemat: $\left.\begin{array}{l} A \sim A_1 \\ B \sim B_2 \\ A \cap B = A_1 \cap B_1 = \phi \end{array}\right\} \longrightarrow A \cup B \sim A_1 \cup B_1$

D-d



$h = f \cup g$

$h : A \cup B \xrightarrow[\text{na}]{1-1} A_1 \cup B_1$

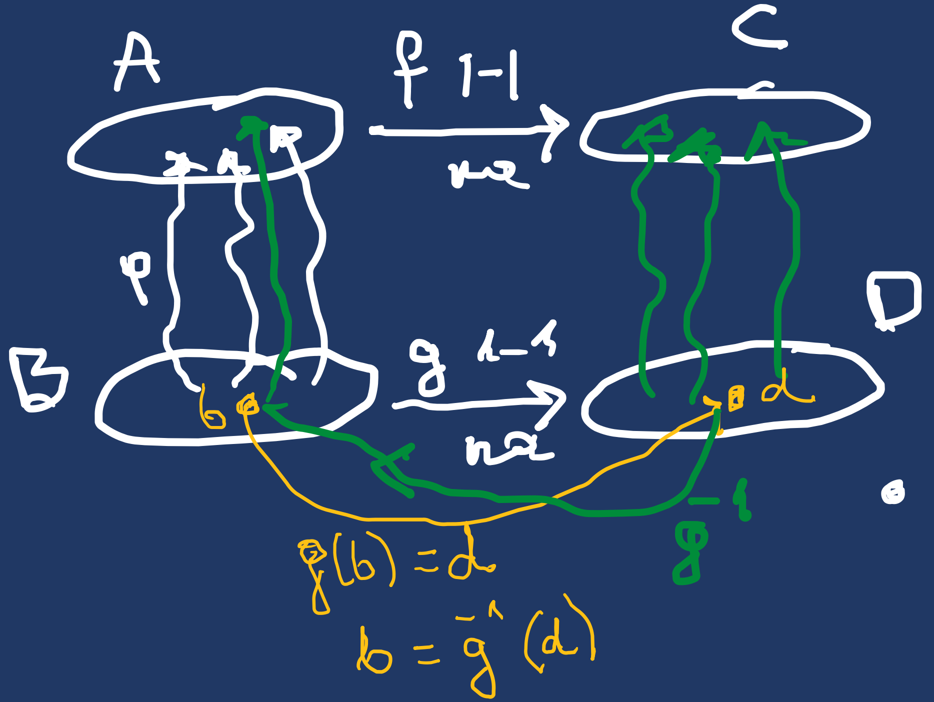
∎

Lemat $\left.\begin{matrix} A \sim A_1 \\ B \sim B_1 \end{matrix}\right] \longrightarrow A \times B \sim A_1 \times B_1$

To jest było.

Lemat $\left.\begin{matrix} A \sim C \\ B \sim D \end{matrix}\right] \longrightarrow A^B \sim C^D$

D-d



Definujemy dla $\varphi \in A^B$

$$\Phi(\varphi) = f \circ \varphi \circ g^{-1}$$

$$\Phi : A^B \to C^D$$

- $\Phi(\varphi) = \Phi(\eta)$

$$f \circ \varphi \circ g^{-1} = f \circ \eta \circ g^{-1} \quad / \circ g$$

$$f \circ \varphi \circ g^{-1} \circ g = f \circ \eta \circ \underbrace{g^{-1} \circ g}_{id_B}$$

$$f\circ\varphi = f\circ\eta \qquad / f^{-1}\circ$$

$$\underbrace{f^{-1}\circ f}_{id_A}\circ\varphi = f^{-1}\circ f\circ\eta$$

$$\varphi = \eta \quad \blacksquare$$

Zadanie: pok. że to jest "na". $\aleph_0$

Ⓟ

$$3+\aleph_0 = |(\{0,1,2\}\times\{0\})\cup(\mathbb{N}\times\{1\})| =$$

$$= |\{0,1,2\}\cup\{3,4,5,6,\dots\}| = |\mathbb{N}| = \aleph_0$$

$$n+\aleph_0 = \aleph_0$$

FAKT: $\kappa+\lambda = \lambda+\kappa$

$\kappa\cdot\lambda = \lambda\cdot\kappa$

$$3+2 = |(\{0,1,2\}\times\{0\})\cup(\{0,1\}\times\{1\})|$$

$$= |\{0,1,2\}\cup\{3,4\}| = |\{0,1,2,3,4\}| = 5 \quad \blacksquare$$

(P) $\aleph_0 + \aleph_0 = |(\mathbb{N}\times\{0\}) \cup (\mathbb{N}\times\{1\})|$

$= |\{\dots,-3,-2,-1\} \cup \{0,1,2,\dots\}| = |\mathbb{Z}| = \aleph_0$

(P) $\aleph_0 = 1\cdot\aleph_0 \leq 2\cdot\aleph_0 \leq 3\cdot\aleph_0 \leq \dots \leq \aleph_0\cdot\aleph_0$

$= |\mathbb{N}\times\mathbb{N}| = |\mathbb{N}| = \aleph_0$

$\aleph_0 \leq 1\cdot\aleph_0 \leq 2\cdot\aleph_0 \leq \dots \leq \aleph_0\cdot\aleph_0 \leq \aleph_0$

$\aleph_0 \leq 3\cdot\aleph_0 \leq \aleph_0$ $\rightsquigarrow$ thm. Cant. Bernst

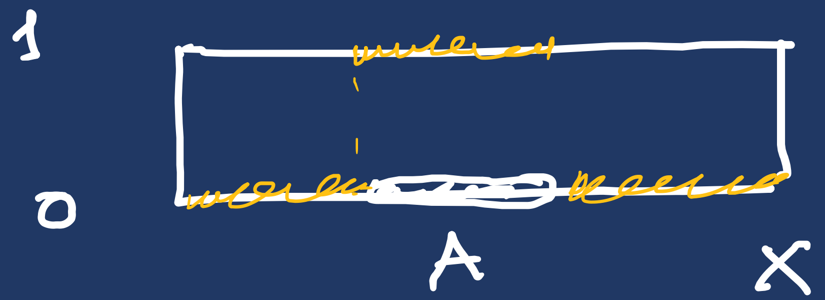
$3\cdot\aleph_0 = \aleph_0$

$\aleph_0 = 1\cdot\aleph_0 = 2\cdot\aleph_0 = \dots = \aleph_0\cdot\aleph_0$

Lemat. $P(X) \sim \{0,1\}^X$

D-d. Dla $A \in P(X)$, czyli $A \subseteq X$, określamy

$$1_A(x) = \begin{cases} 1 : & x \in A \\ 0 : & x \in X \setminus A \end{cases}$$



mapa bitowa zbioru A

Kładziemy

$$\Phi(A) = 1_A .$$

wtedy: $\Phi : P(X) \to \{0,1\}^X$

TO JEST BIJEKCJA:

$$1_A^{-1}(\{1\}) = A$$

ZADANIE

Wniosek:

$$2^{\aleph_0} = \left|\{0,1\}^{\mathbb{N}}\right| \overset{\text{Lemat}}{=} |P(\mathbb{N})| \underset{\text{tw Cantora}}{>} |\mathbb{N}| = \aleph_0$$

$$\boxed{\aleph_0 < 2^{\aleph_0}}$$

CEL DALSZYCH ROZ: $\boxed{2^{\aleph_0} = \mathfrak{c}}$

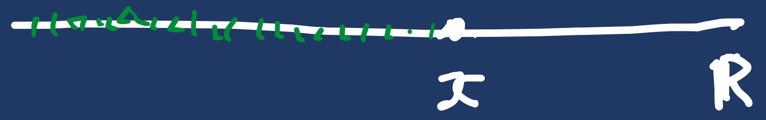
czyli

$$\underset{\underset{|\mathbb{Q}|}{\parallel}}{|\mathbb{N}|} < |\mathbb{R}|$$

FAKT: $|\mathbb{R}| \leq 2^{\aleph_0} = |\{0,1\}^{\mathbb{N}}| = |\{0,1\}^{\mathbb{Q}}| = |P(\mathbb{Q})|$

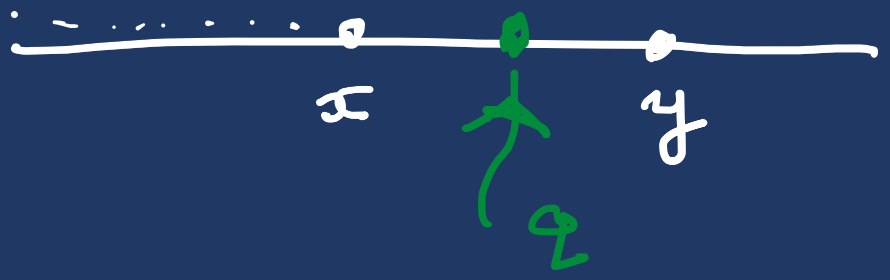
D-d. $\varphi: \mathbb{R} \longrightarrow P(\mathbb{Q})$

$\varphi(x) = \{q \in \mathbb{Q} : q < x\}$



Pok. że $\varphi: \mathbb{R} \xrightarrow{1-1} P(\mathbb{Q})$.

Zał. że $x < y$, $x, y \in \mathbb{R}$



Z gęstości $\mathbb{Q}$ w $\mathbb{R}$ wynika istnienie $q \in \mathbb{Q}$ t. że $x < q < y$

Wtedy: 1) $q \notin \varphi(x)$

2) $q \in \varphi(y)$ więc $\varphi(x) \neq \varphi(y)$

$$\underset{\mathfrak{c}}{\overset{}{|\mathbb{R}|}} \leq \underset{2^{\aleph_0}}{|P(\mathbb{Q})|}$$

ZATEM $\varphi: \mathbb{R} \xrightarrow{1-1} P(\mathbb{Q}) \Rightarrow |\mathbb{R}| \leq |P(\mathbb{Q})|$, gdzie $\mathfrak{c} = |\mathbb{R}|$, $2^{\aleph_0} = |P(\mathbb{Q})|$

MAMY: $\mathfrak{c} \leq 2^{\aleph_0}$.

FAKT: $2^{\aleph_0} \leq \mathfrak{c}$

<u>D-d</u>. Zbudujemy $\varphi : \{0,1\}^{\mathbb{N}} \xrightarrow{1-1} \mathbb{R}$.

Definiujemy

$$\varphi(x) = \sum_{i=0}^{\infty} \frac{x_i}{3^i} \quad \left(= \lim_{N\to\infty} \sum_{i=0}^{N} \frac{x_i}{3^i} \right)$$

Uwaga: 1) to jest szereg o wyrazach ≥ 0.

2) $$\sum_{i=0}^{\infty} \frac{x_i}{3^i} \leq \sum_{i=0}^{\infty} \frac{1}{3^i} = \sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} < \infty$$

2) Weźmy $x, y \in \{0,1\}^{\mathbb{N}}$ t. że $x \neq y$.

Ustalmy $\alpha = \min\{i \in \mathbb{N} : x_i \neq y_i\}$.

Wtedy $x_\alpha, y_\alpha \in \{0,1\}$, $x_\alpha \neq y_\alpha$.

Możemy zał. że $x_\alpha = 0$ i $y_\alpha = 1$.

$$\varphi(x) = \sum_{i=0}^{\alpha-1} \frac{x_i}{3^i} + \frac{\overset{0}{x_\alpha}}{3^\alpha} + \sum_{i>\alpha} \frac{x_i}{3^i} \leq \sum_{i=0}^{\alpha-1} \frac{x_i}{3^i} + \sum_{i=\alpha+1}^{\infty} \frac{1}{3^i}$$

$$= \sum_{i=0}^{\alpha-1} \frac{y_i}{3^i} + \frac{1}{3^{\alpha+1}} \sum_{i=0}^{\infty} \frac{1}{3^i} = \sum_{i=0}^{\alpha-1} \frac{y_i}{3^i} + \frac{1}{3^{\alpha+1}} \cdot \underbrace{\frac{1}{1-\frac{1}{3}}}_{\frac{3}{2}}$$

$$= \sum_{i=0}^{\alpha-1} \frac{y_i}{3^i} + \frac{1}{3^{\alpha+1}} \cdot \frac{3}{2} = \sum_{i=0}^{\alpha-1} \frac{y_i}{3^i} + \frac{1}{2} \frac{y_\alpha}{3^\alpha} < \sum_{i=0}^{\alpha-1} \frac{y_i}{3^i} + \frac{y_\alpha}{3^\alpha} \leq$$

$$< \sum_{l=0}^{\alpha-1} \frac{y_l}{3^l} + \frac{y_\alpha}{3^\alpha} + \sum_{i>\alpha} \frac{y_i}{3^i} = \varphi(y)$$

Zatem $\varphi(x) < \varphi(y)$.

Zatem $\varphi : \{0,1\}^{\mathbb{N}} \xrightarrow{1-1} \mathbb{R}$

Zatem $|\{0,1\}^{\mathbb{N}}| \leq |\mathbb{R}|$ $\boxtimes$

TW $\boxed{2^{\aleph_0} = \mathfrak{c}}$

Q: Ile jest punktów na płaszczyźnie?

czyli:

$$\mathfrak{c}\cdot\mathfrak{c} = ?$$

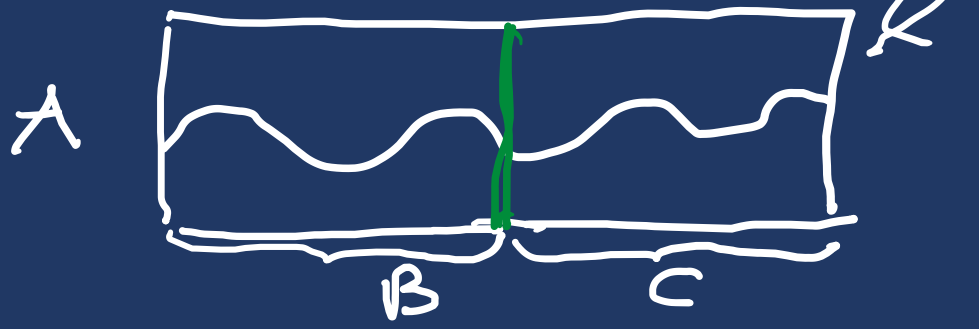
$$|\mathbb{R}\times\mathbb{R}| = \mathfrak{c}\cdot\mathfrak{c}$$

Lemat. Zał. że $B\cap C=\emptyset$. wtedy

$$A^{B\cup C} \sim A^{B}\times A^{C}$$

D-d

$f\in A^{B\cup C}$

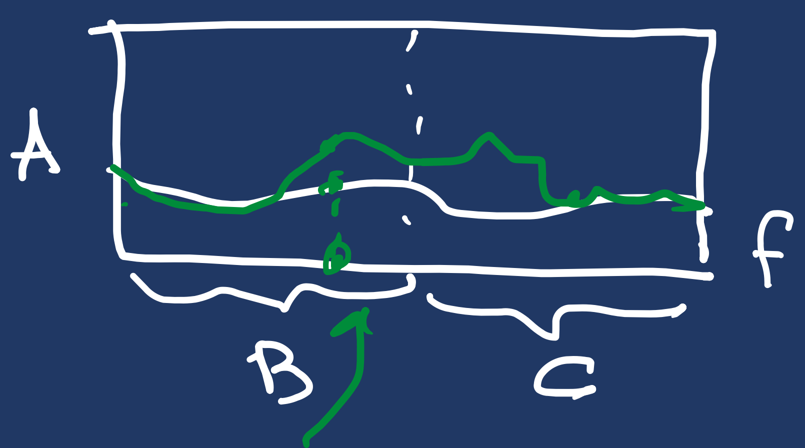


Dla $f\in A^{B\cup C}$ określamy

$$\Phi(f) = (f\restriction_B, f\restriction_C) \in A^{B}\times A^{C}.$$

CLAIM: Φ jest bijekcją.

• 1-1



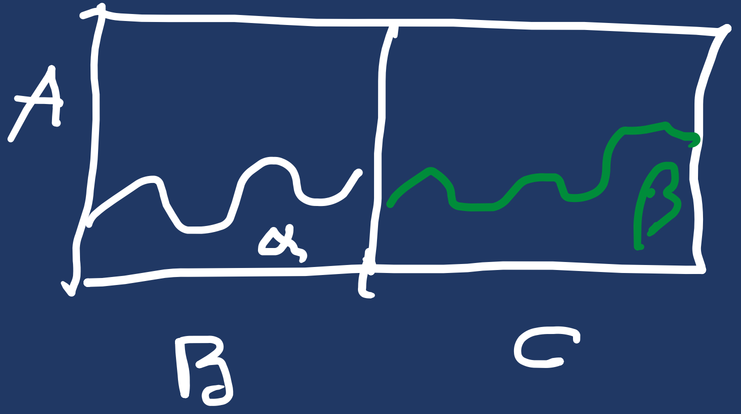
zał. że $f, g \in A^{B\cup C}$ i $f \neq g$

Jest $x \in B \cup C$ t.że $f(x) \neq g(x)$

• $x \in B \setminus C$: wtedy $f\restriction B \neq g\restriction B$

więc $\underbrace{(f\restriction B, f\restriction C)}_{=\,\Phi(f)} \neq \underbrace{(g\restriction B, g\restriction C)}_{=\,\Phi(g)}$

• $x \in C \setminus B$ podobnie

• 

$(\alpha, \beta) \in A^B \times A^C$

wtedy $f = \alpha \cup \beta : B \cup C \to A$

$\Phi(f) = (\alpha, \beta)$.

FAKT: $|[0,\infty)| = \mathfrak{c}$

$$(0,1) \subseteq [0,\infty) \subseteq \mathbb{R}$$

$$\mathfrak{c} \leq |[0,\infty)| \leq \mathfrak{c}$$

FAKT: $|(-\infty,0)| = \mathfrak{c}$

$$\mathfrak{c} \cdot \mathfrak{c} = 2^{\aleph_0} \cdot 2^{\aleph_0} = \left| \{0,1\}^{\{\ldots,-3,-2,-1\}} \times \{0,1\}^{\mathbb{N}} \right|$$

$$= \left| \{0,1\}^{\mathbb{Z}} \right| = 2^{|\mathbb{Z}|} = 2^{\aleph_0} = \mathfrak{c}$$

$$\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$$

P. $\mathfrak{c} = 1\cdot\mathfrak{c} \leqslant 2\cdot\mathfrak{c} \leqslant \ldots \leqslant \aleph_0\cdot\mathfrak{c} \leqslant \mathfrak{c}\cdot\mathfrak{c} = \mathfrak{c}$

$$\mathfrak{c} = 1\cdot\mathfrak{c} = 2\cdot\mathfrak{c} = \ldots = \aleph_0\cdot\mathfrak{c} = \mathfrak{c}\cdot\mathfrak{c} = \mathfrak{c}.$$

P $|\mathbb{R}^3| = |(\mathbb{R}\times\mathbb{R})\times\mathbb{R}| = |\mathbb{R}\times\mathbb{R}|\cdot|\mathbb{R}| = \mathfrak{c}\cdot\mathfrak{c} = \mathfrak{c}$

$$\mathfrak{c}^3 = \mathfrak{c}$$

WNIOSEK. $(\forall n\in\mathbb{N}^+)(\mathfrak{c}^n = \mathfrak{c})$ czyli

$$(\forall n\in\mathbb{N}^+)(|\mathbb{R}^n| = |\mathbb{R}|).$$

W CZWARTEK: $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$