Matematyka dyskretna : WSTĘP

Zał. że $|A|, |B| < \infty$.

- $A \cap B = \emptyset \rightarrow |A \cup B| = |A| + |B|$
- $|A \times B| = |A| \cdot |B|$
- $|A^B| = |A|^{|B|}$

działania na l. naturalnych

$$\begin{cases} n+0 = n \\ n+(m+1) = (n+m)+1 \end{cases}$$

$$\begin{cases} n \cdot 0 = 0 \\ n \cdot (m+1) = n \cdot m + n \end{cases}$$

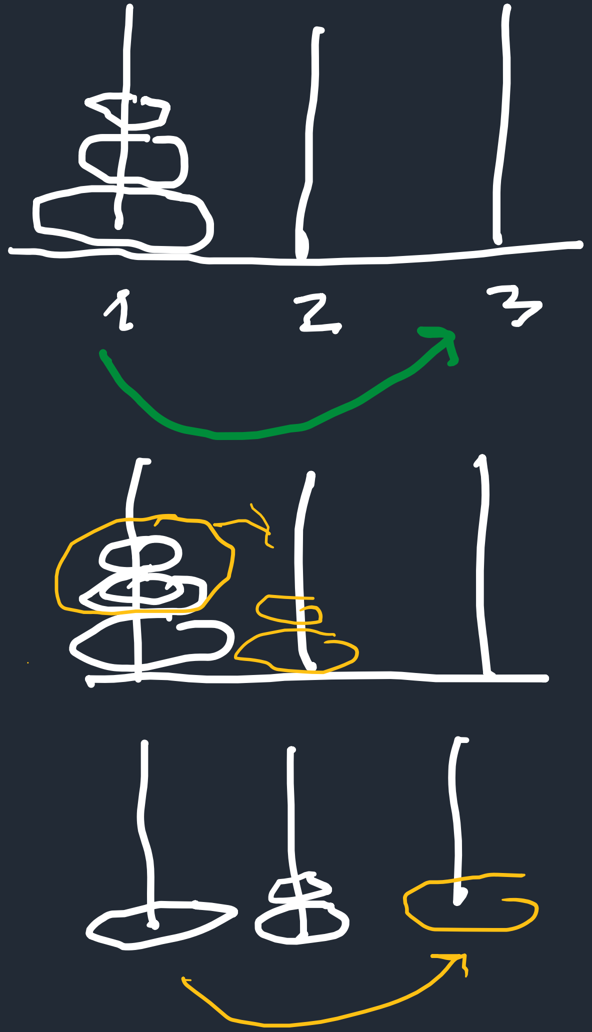
$$\begin{cases} n^0 = 1 \\ n^{m+1} = n^m \cdot n \end{cases}$$

Tw o ZGODNOŚCI

arytmetyki mocy zbiorów skończ. z arytmet. l. naturalnych.

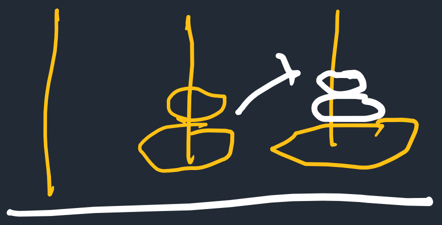
Dowody : indukcja matematyczna.

Przykład. Problem wież z Hanoi



Zasada: nigdy większy nie może leżeć na mniejszym

```
H(n, i, j) {
  if(n==1) { i → j }
  else {
    k = 6-(i+j)
    H(n-1, i, k)
    { i → j }
    H(n-1, k, j)
  }
}
```



T_n = l. przestawień wykon. przez ten algorytm.

$$\begin{cases} T_1 = 1 \\ T_{n+1} = 2 \cdot T_n + 1 \end{cases}$$

n	1	2	3	4	5	6	...
T_n	1	3	7	15	31	63	

HIPOTEZA : $T_n = 2^n - 1$ ← to jest prawdziwe

D-d. 1) $T_1 = 1 = 2^1 - 1$ ok

2) załóż. że dla n jest ok.

$$T_{n+1} = 2 \cdot T_n + 1 \underset{\text{z. ind.}}{=} 2 \cdot (2^n - 1) + 1 = 2^{n+1} - 2 + 1 =$$

$$= 2^{n+1} - 1 \;\blacksquare$$

Sposób wyprowadzenia wzoru

$$\begin{cases} T_1 = 1 \\ T_{n+1} = 2 \cdot T_n + 1 \end{cases}$$

$$t_n = \frac{T_n}{2^n} \quad ; \quad t_1 = \frac{1}{2} \; ; \; t_{n+1} = \frac{T_{n+1}}{2^{n+1}} = \frac{2 \cdot T_n + 1}{2^{n+1}}$$

$$= \frac{T_n}{2^n} + \frac{1}{2^{n+1}} = t_n + \frac{1}{2^{n+1}}$$

$$t_4 = t_3 + \frac{1}{2^4} = \left(t_2 + \frac{1}{2^3}\right) + \frac{1}{2^4} = t_2 + \left(\frac{1}{2^2} + \frac{1}{2^3}\right) =$$

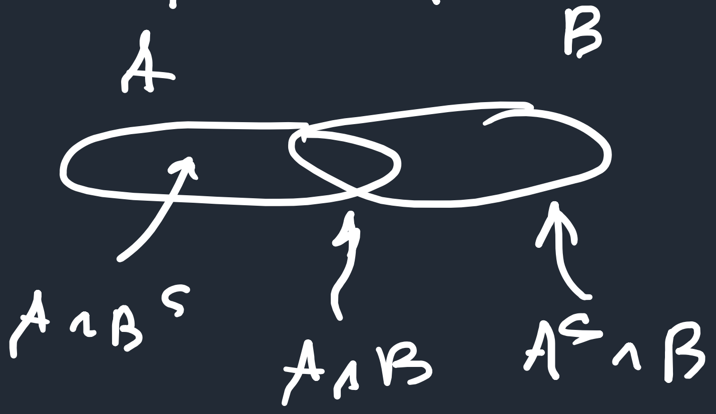
$$= t_1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^4}$$

$$t_n = \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} = \frac{1}{2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}}\right) = \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}$$

$$= 1 - \left(\frac{1}{2}\right)^n \quad ; \quad T_n = 2^n \cdot t_n = 2^n - 1 \quad \blacksquare$$

FAKT: $|A \cup B| = |A| + |B| - |A \cap B|$

Zadanie: zrobić to



UOGÓLNIENIE:

$$|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C| =$$

$$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)| =$$

$$= |A| + |B| + |C| - |A \cap B| - \left(|A \cap C| + |B \cap C| - |A \cap C \cap B \cap C| \right)$$

$$= |A| + |B| + |C| - \left(|A \cap B| + |A \cap C| + |B \cap C| \right) + |A \cap B \cap C| \quad \blacksquare$$

Oznaczenia: • $[n] = \{1, \dots, n\}$ // $[0] = \emptyset$

• $P_+(A) = P(A) \setminus \{\emptyset\}$.

• $\mathcal{A} = \{A_1, \dots, A_n\}$, $T \subseteq [n]$, $T \neq \emptyset$

$$A_T = \bigcap_{i \in T} A_i \text{.} \qquad // A_{\{k\}} = A_k$$

Tw (Zasada włączania - wyłączania)

$\mathcal{A} = \{A_1, \dots, A_n\}$:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{T \in P_+([n])} (-1)^{|T|+1} |A_T| \text{.}$$

Oznaczenie: $[n]^k = \{A \subseteq [n] : |A| = k\}$.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{T \in P_+([n])} (-1)^{|T|+1} \cdot |A_T| =$$

inne sformułowanie

$$= \sum_{k=1}^{n} (-1)^{k+1} \sum_{T \in [n]^k} |A_T| . \qquad (*)$$

ZWW : Principle of Inclusion-Exclusion

PIE - principle

Zadanie: zastosuj (*) dla $n=2,3$

D-d. Dla $n=2$ to ok. $|A\cup B| = |A|+|B|-|A\cap B|$

Załóż. że dla n jest ok.

$$\left|\bigcup_{i=1}^{n+1} A_i\right| = \left|\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right| = \left|\bigcup_{i=1}^{n} A_i\right| + |A_{n+1}| - \left|\left(\bigcup_{i=1}^{n} A_i\right)\cap A_{n+1}\right|$$

$$= \sum_{T\in P_+([n])} (-1)^{|T|+1} |A_T| + |A_{n+1}| - \sum_{T\in P_+([n])} (-1)^{|T|+1} |B_T| =$$

$$= \sum_{T\in P_+([n])} (-1)^{|T|+1} |A_T| + |A_{n+1}| - \sum_{T\in P_+([n])} (-1)^{|T|+1} |A_{T\cup\{n+1\}}|$$

$$= \sum_{T} (-1)^{|T|+1} |A_T| + |A_{n+1}| \leftarrow A_{\emptyset\cup\{n+1\}}$$

$$+ \sum_{T\in P_+([n])} (-1)^{|T\cup\{n+1\}|+1} |A_{T\cup\{n+1\}}|$$

$$= (*)$$

$T\subseteq [n]$, $T\neq\emptyset$

$$B_T = \bigcap_{i\in T} B_i = A_{T\cup\{n+1\}}$$

$$\left(\bigcup_{i=1}^{n} A_i\right)\cap A_{n+1} = \bigcup_{i=1}^{n} \underbrace{(A_i\cap A_{n+1})}_{B_i}$$

również $T=\emptyset$

$$P_+([n+1]) = \underbrace{P_+([n])}_{\text{bez } n+1} \cup \underbrace{\{T \cup \{n+1\} : T \subseteq [n]\}}_{\text{zawiera } n+1}$$

$$(*) = \sum_{T \in P_+([n])} (-1)^{|T|+1} |A_T| + \sum_{T \in P([n])} (-1)^{|T \cup \{n+1\}|+1} |A_{T \cup \{n+1\}}|$$

$$= \sum_{T \in P_+([n+1])} (-1)^{|T|+1} |A_T| \quad \boxtimes$$

Ozn. $\mathrm{Sym}(A) = \{\pi \in A^A : \pi : A \xrightarrow[\text{na}]{1-1} A\}$

$\mathrm{Sym}_n = \mathrm{Sym}([n])$

$\mathrm{Sym}(\emptyset) = \{\emptyset\}$.

FAKT: $|\mathrm{Sym}_n| = n!$

$$\begin{cases} 0! = 1 \\ n! = \prod_{i=1}^{n} i \end{cases}$$

D-d. indukcja po n.

Dla $n = 0, 1$! ok.

$\sigma \in S_n$: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \longleftrightarrow [3, 2, 1, 4]$

$|S_n| = n!$

Jak zrobić z $\sigma \in S_n$ perm. $n+1$ elem.

$\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$

$$\left.\begin{array}{l} \sigma_0 = [n+1, \sigma_1, \sigma_2 \ldots \sigma_n] \\ \sigma_1 = [\sigma_1, n+1, \sigma_2, \ldots \sigma_n] \\ \vdots \\ \sigma_n = [\sigma_1, \sigma_2, \ldots, \sigma_n, n+1] \end{array}\right\} n+1 \text{ możliwości}$$

$$Sym_{n+1} = \bigcup_{\sigma \in Sym_n} \{\sigma_i : i = 0 \ldots n\}$$

$$|Sym_{n+1}| = |Sym_n| \cdot (n+1) = n! \cdot (n+1) = (n+1)!$$

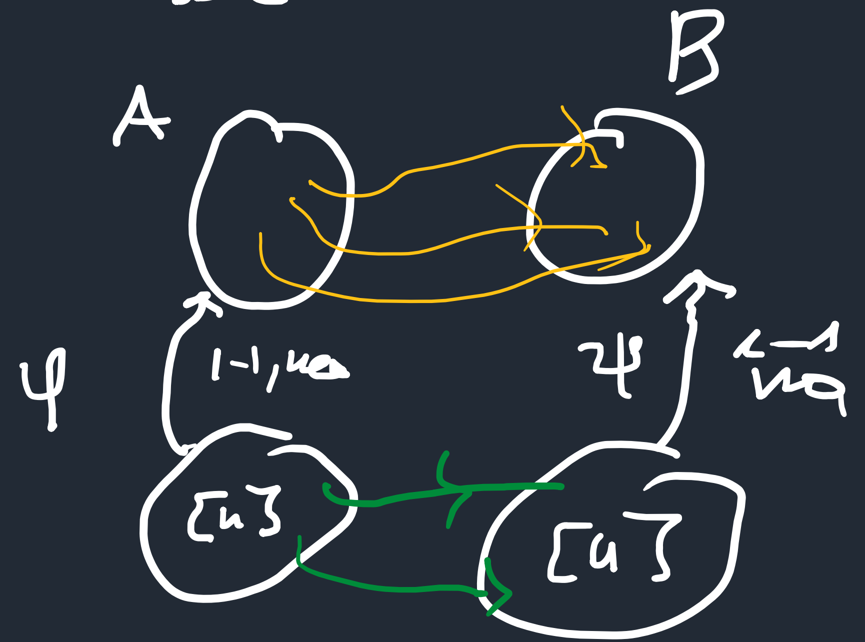
$\boxed{}$

UWAGA: $|A| = |B|$

$$\mathrm{Bij}(A,B) = \{ f \in B^A : f: A \xrightarrow[\text{na}]{1-1} B \}$$

$$|\mathrm{Bij}(A,B)| = |A|!$$

$$F(f) = \psi^{-1} \circ f \circ \varphi$$



$n = |A|$

Zadanie:
Sprawdzić, że

$$F: \mathrm{Bij}(A,B) \xrightarrow[\text{na}]{1-1} \mathrm{Sym}_n$$

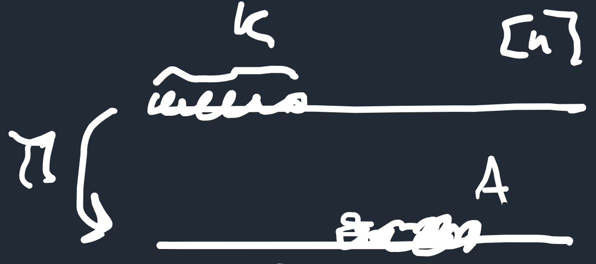
• $[n]^k = \{A \subseteq [n] : |A| = k\}$

Symbol Newtona

Tw. $|[n]^k| = \binom{n}{k} \left(= \frac{n!}{k!\,(n-k)!} \right)$

D-d:

$$Sym_n = \bigcup_{A \in [n]^k} \{\pi \in S_n : \pi([k]) = A\}$$

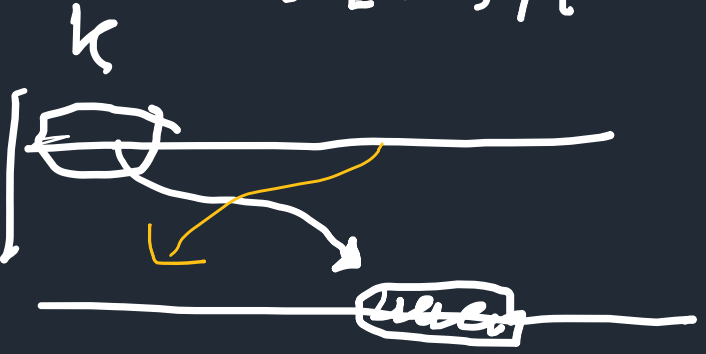


$\pi([k]) = A$

$|\pi([k])| = k$

$$|\{\pi \in S_n : \pi([k]) = A\}|$$

$$= |\{\varphi \cup \psi : \varphi \in Bij([k], A) \wedge \psi \in Bij([n] \setminus [k], [n] \setminus A)\}|$$



$$= k!\,(n-k)!$$

$$n! = |\mathrm{Sym}_n| = \sum_{A \in [n]^k} |\{\pi \in S_n : \pi([k]) = A\}|$$

$$= \sum_{A \in [n]^k} k!\,(n-k)! = |[n]^k| \cdot k!\,(n-k)!$$

$$\boxed{|[n]^k| = \frac{n!}{k!\,(n-k)!} \overset{\text{def}}{=} \binom{n}{k}}$$

!!!

(P)

```
FOR I=1 TO N
  FOR J=I+1 TO N
    OP(I,J)
```

Ile razy wykonywane jest OP?

$N = 4$

Q: OP(I,J) = "print i, j"

$\{1,2\}$
$\{1,3\}$
$\{1,4\}$
$\{2,3\}$
$\{2,4\}$
$\{3,4\}$

$$L_N = |[N]^2| = \binom{N}{2} = \frac{N(N-1)}{2}$$

$$= \frac{1}{2}(N^2 - N)$$

$$\lim_{N\to\infty} \frac{L_N}{\frac{1}{2}N^2} = \lim_{N\to\infty} \frac{N(N-1)}{N^2} = \lim_{N\to\infty}\left(1-\frac{1}{N}\right) = 1$$

$a_n, b_n > 0$

$$(a_n) \sim (b_n) \equiv \lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

WNIOSEK : $\boxed{L_n \sim \frac{1}{2} n^3}$

asymp. równowa.