NIEPORZĄDKI

$\pi \in \mathrm{Sym}(n) \equiv (\forall i \in [n])(\pi(i) \neq i)$

CEL: Ile jest nieporządków w $\mathrm{Sym}(n)$?

Term: i jest fix-point'em π $\equiv \pi(i) = i$
(punktem stałym)

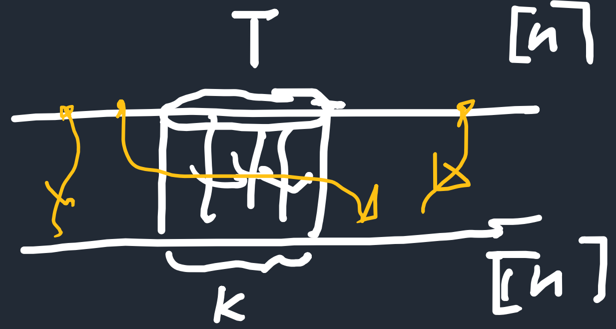
$F_i = \{\pi \in S_n : \pi(i) = i\}$

$\left|\bigcup_{i=1}^{n} F_i\right| = ?$

$F_i \cap F_j = \{\pi \in S_n : \pi(i) = i \wedge \pi(j) = j\}$

Dla $T \subseteq [n]$ określamy

$$F_T = \bigcap_{i\in T} F_i \quad \left(= \{\pi \in S_n : (\forall i \in T)(\pi(i) = i)\}\right)$$



$$\left|\bigcup_{i=1}^{n} F_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{T\in[n]^k} |F_T| =$$

$$|[n]\setminus T| = n - |T|$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \sum_{T\in[n]^k} (n-|T|)! =$$

$$F_T = \{ id_T \cup \varphi : \varphi : [n]\setminus T \xrightarrow[\text{na}]{1-1} [n]\setminus T \}$$

$$|F_T| = (n-|T|)!$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)! =$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!\,(n-k)!} (n-k)! = \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!}$$

$$= n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}.$$

$$N_n = \{\pi \in S_n : (\forall i \in [n])(\pi(i) \neq i)\}$$

$$|N_n| = |S_n| - \left|\bigcup_{i=1}^{n} F_i\right| = n! - n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}$$

$$= n!\left(1 + \sum_{k=1}^{n} \frac{(-1)^{k}}{k!}\right)$$

$$\boxed{\frac{|N_n|}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}}$$

ANALIZA :
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

WNIOSEK : $\boxed{\lim_{n\to\infty} \frac{|N_n|}{n!} = \frac{1}{e}}$ $\left(\approx \frac{1}{3}\right)$

Przestrzenie probabilistyczne dyskretne

- mamy zbiór przeliczalny Ω
  (zbiór zdarzeń elementarnych)
- mamy $p : \Omega \longrightarrow [0,1]$ t.że
  $$\sum_{\omega \in \Omega} p(\omega) = 1$$
  ($p(\omega)$ = prawd. wylosowania ω)
- dla $A \subseteq \Omega$ (zdarzenia) określamy
  $$Pr(A) = \sum_{\omega \in A} p(\omega).$$

$(\Omega, \mathcal{P}(\Omega), \Pr) \leftarrow$ pr. prob. dyskr.

- $\Pr(\emptyset) = 0$
- $\Pr(\Omega) = \left(\sum_{\omega \in \Omega} p(\omega)\right) = 1$
- $0 \leq \Pr(A) \leq 1$
- $A, B \subseteq \Omega,\ A \cap B = \emptyset \longrightarrow$

$$\Pr(A \cup B) = \sum_{\omega \in A \cup B} p(\omega) = \sum_{\omega \in A} p(\omega) + \sum_{\omega \in B} p(\omega)$$

$$= \Pr(A) + \Pr(B)$$

- wzmocnienie: jeśli $A_n \subseteq \Omega$ są parami rozłączne to

$$Pr\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} Pr(A_n)$$

// bezwz. zbieżn. $\sum_{\omega} p(\omega) = 1$

- wnioski
  - $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

ZADANIE

  - $Pr(A^c) = 1 - Pr(A)$

$A \subseteq \Omega$

BO $1 = Pr(\Omega) = Pr(A \cup A^c) = Pr(A) + Pr(A^c)$

Prawd. kombinatoryczne:

Ω - skończony

$$p(\omega) = \frac{1}{|\Omega|}$$

$$\Pr(A) = \sum_{\omega \in A} p(\omega) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{1}{|\Omega|} \sum_{\omega \in A} 1$$

$$= \frac{|A|}{|\Omega|} .$$

P) $\Omega = \{1, 2, 3, 4\}$

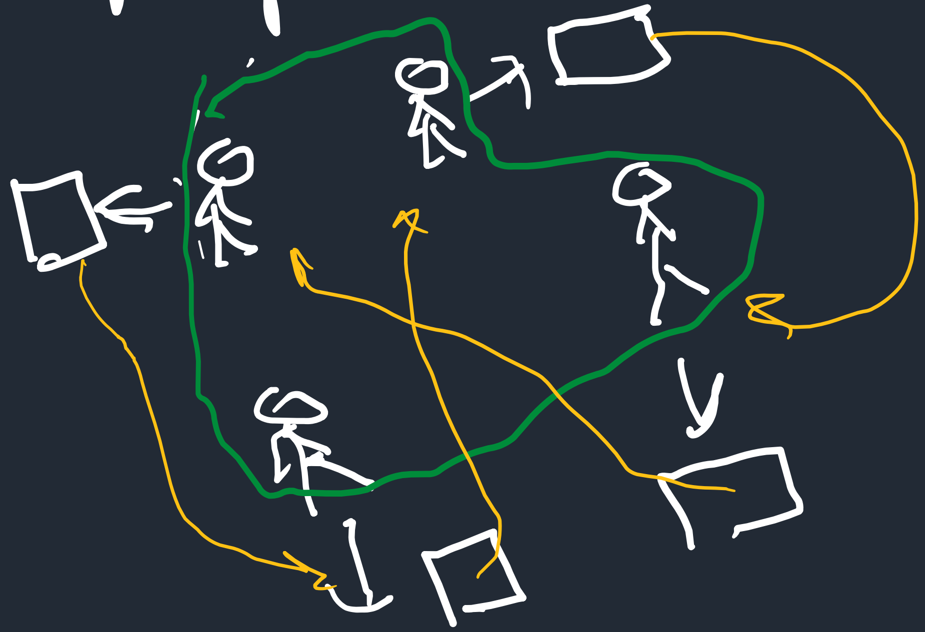
$$p(\omega) = \frac{1}{4}$$

interpr: prawd. wylosowania dow. zdarz. element są takie same

$$\frac{|N_n|}{n!} = Pr(N_n) \qquad \Omega = Sym_n.$$

wniosek: $\boxed{\lim_{n\to\infty} Pr(N_n) = \frac{1}{e}}$

przykład:



permutujemy ankiety

$$Pr[\exists_i\ \pi(i)=i] \simeq 1-\frac{1}{e} \approx \frac{2}{3}$$

WSPÓŁCZYNNIKI DWUMIANOWE

(wsp. Newtona)

$$\binom{n}{k} = \left|[n]^k\right| \quad , \ 0 \le k \le n, \ k, n \in \mathbb{N}.$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

a co się dzieje z $k > n$?

$$\binom{3}{4} = \left|[3]^4\right| = 0$$

$$\binom{0}{0} = \left|[0]^0\right| = 1$$

$$\binom{0}{k} = \begin{cases} 1 : & k = 0 \\ 0 : & k > 0 \end{cases}$$

- nawias logiczny : φ – zdanie logiczne

$$[\varphi] = \begin{cases} 1 : & \varphi \\ 0 : & \neg\varphi \end{cases}$$

$$\binom{0}{k} = [k = 0]$$

- $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$0 \le k \le n$ $$\binom{n}{n-k} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

Tożsamość Pascala:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$[k \geq 1, n \geq 1]$

<u>D-d</u>. $\binom{n}{k} = \left| [n]^{k} \right| =$

$= \left| \{T \subseteq [n] : |T| = k \wedge n \notin T\} \,\dot{\cup}\, \{T \subseteq [n] : |T| = k \wedge n \in T\} \right|$

$= \left| [n-1]^{k} \right| + \left| \{S \cup \{n\} : S \in [n-1]^{k-1}\} \right| = \binom{n-1}{K} + \binom{n-1}{K-1}$

$\blacksquare$

- $n \geq 1$ $\qquad \binom{n}{0} = 1 \qquad \binom{n}{n} = 1$

- $n \geq 1$: $\qquad \binom{n}{1} = \left| [n]^1 \right| = n$

$$\binom{n}{n-1} = \binom{n}{n-(n-1)} = \binom{n}{1} = n.$$

Ulse ge : $\quad \binom{n}{1} = \dfrac{n!}{1!\,(n-1)!} = \dfrac{(n-1)!\,n}{(n-1)!} = n.$

$$\binom{0}{0}$$
$$\binom{1}{0} \binom{1}{1}$$
$$\binom{2}{0} \binom{2}{1} \binom{2}{2}$$
$$\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3}$$

. . . .

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{n}{k} = \underline{\binom{n-1}{k}} + \underline{\binom{n-1}{k-1}}$$

0:	1						→ 1
1:	1	1					→ 2
2:	1	2	1				→ 4
3:	1	3	3	1			→ 8
4:	1	4	6	4	1		→ 16
5:	1	5	10	10	5	1	→ 32

~~HIPO~~: TW

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

$$2^n = |P([n])| = \left| \biguplus_{k=0}^{n} [n]^k \right| =$$

$$= \sum_{k=0}^{n} \left| [n]^k \right| = \sum_{k=0}^{n} \binom{n}{k} .$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = [n=0]$$

1						1
1	1					0
1	2	1				0
1	3	3	1			0
1	4	6	4	1		0
1	5	10	10	5	1	0

Twzór dwumianowy Newtona

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$$

P

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

P

$$0^n = ((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

$$\|$$

$$[n=0]$$

P

$$(m+1)^n = \sum_{k=0}^{n} \binom{n}{k} m^k .$$