Wzór dwumianowy Newtona.

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$$

$$(x+y)^3 = [(x+y)(x+y)](x+y) = [x\cdot(x+y) + y\cdot(x+y)]\cdot(x+y)$$

$$= [(x\cdot x) + (x\cdot y) + (y\cdot x) + (y\cdot y)]\cdot(x+y)$$

$$= (x\cdot x\cdot x) + (x\cdot y\cdot x) + (y\cdot x\cdot x) + (y\cdot y\cdot x) +$$

$$+ (x\cdot x\cdot y) + (x\cdot y\cdot y) + (y\cdot x\cdot y) + (y\cdot y\cdot y).$$

$T \subseteq [n]$: $\quad e_T = \prod_{k=1}^{n} (\text{if } k \in T \text{ then } x \text{ else } y)$

$T = \{1,3\} \subseteq [4]$

$e_T = x \cdot y \cdot x \cdot y$

$$(x+y)^n = \sum_{T \subseteq [n]} e_T$$

// indukcja

przemienność $\cdot$

$$= \sum_{k=0}^{n} \sum_{T \in [n]^k} e_T$$

$e_T = x^{|T|} \cdot y^{n-|T|}$

$$(x+y)^n = \sum_{k=0}^{n} \sum_{T \in [n]^k} x^k \cdot y^{n-k} \cdot 1$$

$$= \sum_{k=0}^{n} x^k y^{n-k} \sum_{T \in [n]^k} 1 =$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k} \quad \blacksquare$$

Wniosek. W pierścieniu $(R, +, \circ)$ dla dowolnych $x, y \in R$ i $n \geq 0$ jeśli $x \circ y = y \circ x$ to

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

(P) $M_n(K)$ = pierścień macierzy wym. $n \times n$ nad K:

$A, B \in M_n(K)$ i $A \circ B = B \circ A$

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}.$$

(P) $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$

$0^n = (-1+1)^n = \sum_{k=0}^{n} \binom{n}{k}(-1)^k$ $\quad n \geq 0.$

SYMETRIA: $\binom{n}{k} = \binom{n}{n-k}$ $\quad 0 \leq k \leq n,\ n \in \mathbb{N}.$

TROJ. PASC: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ $\quad n \geq 1,\ k \geq 1$

REDUKCJA: $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ $\quad n, k \geq 1$

D-d. $\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!\,(n-k)!}$

$(k-1)+(n-k) = n-1$

$$= \frac{n}{k}\binom{n-1}{k-1} \quad \blacksquare$$

P $\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k \cdot \frac{n}{k}\binom{n-1}{k-1} = n\sum_{k=1}^{n}\binom{n-1}{k-1}$

$$= n\sum_{l=0}^{n-1}\binom{n-1}{l} = n \cdot 2^{n-1}.$$

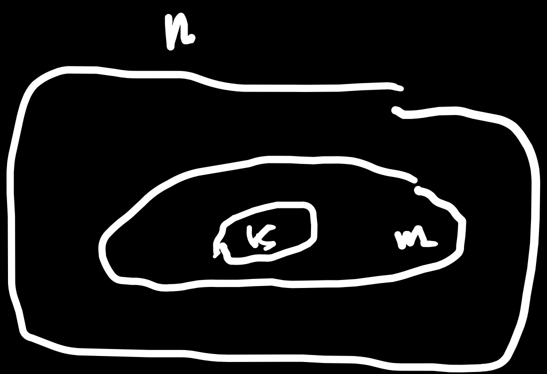
$l = k-1$

zwarta postać

W. $0 \leq k \leq m \leq n$

$(n-m)+(m-k) = n-k$

$$\binom{n}{m}\cdot\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$



D-d

$$\frac{n!}{m!\,(n-m)!}\cdot\frac{m!}{k!\,(m-k)!} =$$

$$= \frac{n!}{k!}\cdot\frac{1}{(n-m)!\,(m-k)!} = \frac{n!}{k!\,(n-k)!}\cdot\frac{(n-k)!}{(n-m)!\,(m-k)!} \quad \square$$

SUMY WSP. DWUMIANOWYCH

1						
1	1	$\boxed{0}$	0	0	0	0
1	2	$\boxed{1}$	0	0	0	0
1	3	$\boxed{3}$	1	0	0	0
1	4	$\boxed{6}$	4	1	0	0
1	5	$\boxed{10}$	10	5	1	0
1	6	15	20	15	6	1

$$\sum_{k \le n} \binom{k}{a} = \binom{n+1}{a+1}$$

Fix a:

Dla $n = 1$: ok

$$\sum_{k \le n+1} \binom{k}{a} = \binom{n+1}{a+1} + \binom{n+1}{a} = \binom{n+2}{a+1} \quad \blacksquare$$

indukcja załatwia sprawę.

$$
\begin{array}{ccccccc}
1 \\
1 & 1 \\
\boxed{1} & 2 & 1 \\
1 & \boxed{3} & 3 & 1 \\
1 & 4 & \boxed{6} & 4 & 1 \\
1 & 5 & 10 & \boxed{10} & 5 & 1 \\
1 & 6 & 15 & \boxed{20} & 15 & 6 & 1
\end{array}
$$

$$\sum_{k\le n}\binom{a+k}{k} \overset{\text{sym}}{=} \sum_{k\le n}\binom{a+k}{a} =$$

$$= \binom{a+n+1}{a+1} \overset{\text{sym}}{=} \binom{a+n+1}{n}$$

$m \geq 1$

$$\boxed{\sum_{0 \leq k \leq m} \binom{n}{k} (-1)^k = \binom{n-1}{m} (-1)^m}$$

$$= (-1)^{m+1} \left(\binom{n-1}{m+1} + \binom{n-1}{m} - \binom{n-1}{m} \right)$$

$$= (-1)^{m+1} \binom{n-1}{m+1}$$

$\blacksquare$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

$n = 6 \longrightarrow$ $1 \quad -5 \quad 10 \quad -10 \quad 5 \quad -1 \quad 0$

D-d:

$$\binom{n-1}{m}(-1)^m + \binom{n}{m+1}(-1)^{m+1} = (-1)^{m+1}\left(\binom{n}{m+1} - \binom{n-1}{m} \right) =$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$ ← wielomian stopnia n

$$(1+x)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} x^k$$

$$\parallel$$

$$(1+x)^n (1+x)^m = \left(\sum_{k=0}^{n} \binom{n}{k} x^k \right) \cdot \left(\sum_{k=0}^{m} \binom{m}{k} x^k \right)$$

$$= \sum_{k=0}^{n+m} x^k \cdot \sum_{\substack{a+b=k \\ a,b \geq 0}} \binom{n}{a} \binom{m}{b}$$

$$\boxed{\binom{n+m}{k} = \sum_{a+b=k} \binom{n}{a}\binom{m}{b}}$$

$$= \sum_{a=0}^{k} \binom{n}{a}\binom{m}{k-a}$$

Toż. Vandermonde – Cauchy'ego

(P) $n = m = k$

$$\binom{2n}{n} = \sum_{a=0}^{n} \binom{n}{a}\binom{n}{n-a} \overset{\text{sym}}{=} \sum_{k=0}^{n} \binom{n}{k}^2$$

TOŻ. CAUCHY'ego.