MULTIZBIORY

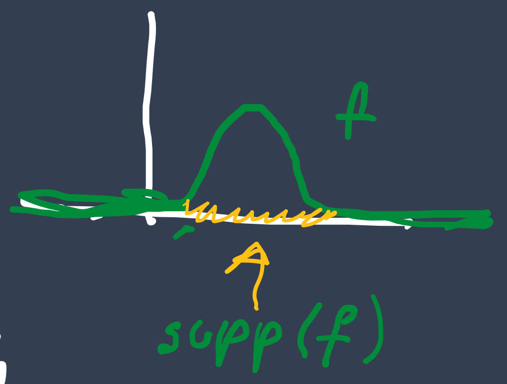
idea: $A = \{\circ, \square, \triangle\}$

$m = (3\cdot\circ + 2\cdot\square + 4\cdot\triangle)$ $\qquad \|m\| = 3+2+4$

X - ustalony zbiór

$f : X \to \mathbb{N}$ $\qquad f(x) = k \equiv$ "x występuje k razy"

$\operatorname{supp}(f) = \{x \in X : f(x) \neq 0\}$



DEF:

$$\mathcal{M}(X) = \{f \in \mathbb{N}^X : |\operatorname{supp}(f)| < \aleph_0\}$$

$m \in \mathcal{M}(X)$: $\|m\| = \sum_{x \in X} m(x) \in \mathbb{N}$.

$f, g \in \mathcal{M}(X)$.

- $(f+g)(x) = f(x) + g(x)$ $\quad$: $\quad f+g \in \mathcal{M}(X)$
- $(f \wedge g)(x) = \min\{f(x), g(x)\}$ $\quad f \wedge g \in \mathcal{M}(X)$
- $(f \vee g)(x) = \max\{f(x), g(x)\}$. $\quad f \vee g \in \mathcal{M}(X)$

$f, g \in \{0,1\}^X$

$$\left.\begin{matrix} \wedge & \longleftrightarrow & \cap \\ \vee & \longleftrightarrow & \cup \end{matrix}\right\} \text{zadanie}$$

Ilość multizbiorów:

$X = \{a, b, c\}$ $|X| = k$

$m \in \mathcal{M}(X)$: $\|m\| = n$

np. $n = 6$:

$6 \cdot a + 0 \cdot b + 0 \cdot c$

$2 \cdot a + 3 \cdot b + 1 \cdot c$

$a < b < c$

$2 \cdot a + 3b + 1c$

$00|000|0$

$3 \cdot a + 0 \cdot b + 3 \cdot c$

$000||000$

$\binom{n+k-1}{k-1}$

ogólniej:

$k-1$

$\underbrace{000\ldots0}_{n_1}|\underbrace{000}_{n_2}|\underbrace{000}_{n_3}|\underbrace{000}_{n_4}|\ldots|\underbrace{0000\ldots0}_{n_k}$

$n_1 + \ldots + n_k = n$

łączna długość: $n + k - 1$

Wniosek. Jeśli $|X| = k$ to

$$\left|\{f \in \mathcal{M}(X) : \|f\| = n\}\right| = \binom{n+k-1}{k-1}$$

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n} =$$

$$= \frac{(n+k-1)^{\underline{n}}}{n!} = \frac{(n+k-1)\cdot\ldots\cdot\overbrace{(n+(k-1)-(n-1))}^{k}}{n!} = \frac{k\cdot(k+1)\cdot\ldots\cdot(k+(n-1))}{n!}$$

$$= \frac{k^{\overline{n}}}{n!} \quad \left(= \left(\!\!\binom{k}{n}\!\!\right) \right)$$

P Ile rozw. naturalnych ma równanie

$$x_1 + x_2 + x_3 + x_4 = 20\ ?$$

$$n = x_1 \cdot 1 + x_2 \cdot 2 + x_3 \cdot 3 + x_4 \cdot 4$$

$$\|n\| = x_1 + \ldots + x_4$$

$$\left|\{f \in M(\{1,2,3,4\}) : \|f\| = 20\}\right| = \binom{20+4-1}{4-1} = \binom{23}{3}.$$

(P) Ile jest rozw. nat. równania

$$x_1 + x_2 + x_3 + x_4 = 20$$

t.że $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 1$, $x_4 \geq 3$?

$y_1 = x_1 - 1 \geq 0$

$y_2 = x_2 - 2 \geq 0$

$y_3 = x_3 - 1 \geq 0$

$y_4 = x_4 - 3 \geq 0$

$$(y_1+1) + (y_2+2) + (y_3+1) + (y_4+3) = 20$$

$$y_1 + y_2 + y_3 + y_4 = 20 - 7 = 13$$

$$\binom{13+4-1}{4-1} = \ldots$$

(P) $\circ\circ\circ,\ \square,\square,\ \triangle$ $\qquad 3\cdot\circ + 2\cdot\square + 1\cdot\triangle$

ile jest tych wariantów ustawień

$\left\{\begin{array}{l} \circ\ \circ\ \circ\ \square\ \square\ \triangle \\ \square\ \square\ \circ\ \circ\ \circ\ \triangle \\ \square\ \circ\ \triangle\ \circ\ \circ\ \square \\ \vdots \end{array}\right.$

$$\left[\begin{array}{l} f:[6]\longrightarrow\{\circ,\square,\triangle\} \\ |f^{-1}(\{\circ\})| = 3 \\ |f^{-1}(\{\square\})| = 2 \\ |f^{-1}(\{\triangle\})| = 1 \end{array}\right.$$

$$f:[6]\longrightarrow\{\circ,\square,\triangle\}$$

$$\downarrow$$

$$a_f = (a_1, a_2, \dots, a_6) \in \{\circ,\square,\triangle\}^6$$

Ustalamy $a \in \{\circ, \square, \triangle\}^6$ $\qquad a = (\circ, \circ, \circ, \square, \square, \triangle)$

Dla $\pi \in S([6]) = S_6$ $\qquad \pi : [6] \xrightarrow[\text{na}]{1-1} [6]$

$$\pi(a) = (a_{\sigma(1)}, \dots, a_{\sigma(6)})$$

$$\pi(a) = \sigma(a) \equiv (\forall i = 1..6)\left(a_{\pi(i)} = a_{\sigma(i)}\right)$$

$j = \pi(i)$

$i = \pi^{-1}(j)$

$\pi, \sigma \in S_6$

$$\equiv (\forall j)\left(a_j = a_{\sigma(\pi^{-1}(j))}\right)$$

$$\equiv (\forall j)\left(a_j = a_{\sigma \circ \pi^{-1}(j)}\right)$$

$\rho = \sigma \circ \pi^{-1}$

$$\equiv (\forall j)\left(a_j = a_{\rho(j)}\right)$$

$$Fix(a) = \{\rho \in S_6 : (\forall j)\left(a_j = a_{\rho(j)}\right)\}$$

$$\equiv \sigma \circ \pi^{-1} \in Fix(a).$$

$$\pi(a) = \sigma(a) \equiv \sigma \circ \pi^{-1} \in \mathrm{Fix}(a)$$

$\pi \circ \sigma^{-1}$

S_6

$\mathrm{Fix}(a)$

- $\mathrm{Fix}(a) \triangleleft S_6$

$$\equiv \sigma \in \mathrm{Fix}(a) \circ \pi$$

warstwa podgr. $\mathrm{Fix}(a)$

$$|\mathrm{Fix}(a)| = 3! \cdot 2! \cdot 1!$$

GDP: $\dfrac{|S_6|}{|\mathrm{Fix}(a)|} = \dfrac{6!}{3!\,2!\,1!}$

OGÓLNIEJ ! $m = a_1 \cdot 1 + \dots + a_k \cdot k$

liczba różnych permutacji m

$$\|m\| = a_1 + \dots + a_k$$

$$= \frac{(a_1 + \dots + a_k)!}{(a_1!) \cdot \ldots \cdot (a_k)!} = \binom{a_1 + \dots + a_k}{a_1, \dots, a_k}$$

WSPÓŁCZ. MULTIMIANOWY :

$$\binom{n}{a_1, a_2, \dots, a_k} = \frac{n!}{(a_1)! \cdot \ldots \cdot (a_k)!}$$

gdzie $n = a_1 + \dots + a_k$

UWAGA

$$\binom{n+m}{n, m} = \binom{n+m}{n}$$

ZADANIE

$$(x_1+x_2+\dots+x_k)^n = \sum_{a_1+\dots+a_k=n} \binom{n}{a_1,\dots,a_k} x_1^{a_1}\cdot\ldots\cdot x_k^{a_k},$$

$$\binom{a+b+c}{a,b,c} = \frac{(a+b+c)!}{a!\cdot b!\, c!} = \frac{(a+b+c)!}{a!\,(b+c)!}\cdot\frac{(b+c)!}{b!\,c!} =$$

$$= \binom{a+b+c}{a}\cdot\binom{b+c}{b}$$

HINT

$$(x_1+x_2+x_3)^n = (x_1+(x_2+x_3))^n = \sum_a \binom{n}{a} x_1^a (x_2+x_3)^{n-a} =$$

$$= \sum_a \sum_b \dots\dots .$$

FUNKCJE TWORZĄCE

n-fix : $\sum \binom{n}{k} x^k = (1+x)^n$

$\begin{cases} \text{weź } f(x) = (1+x)^n \\ \text{rozwiń w szereg Taylora.} \end{cases}$

$$\sum_n (1+x)^n y^n = \sum_{n \geq 0} ((1+x)y)^n = \frac{1}{1-(1+x)y}$$

$$\parallel$$

$$\sum_n \left(\sum_k \binom{n}{k} x^k \right) y^n = \sum_{n,k} \binom{n}{k} x^k y^n$$

$|x| < 1$

$|y| < \frac{1}{2}$

DEF. Niech $a : \mathbb{N} \longrightarrow \mathbb{C}$. Funkcję tworzącą ciągu a nazywamy szereg

$$F_a(x) = \sum_{n \geq 0} a_n x^n .$$

Hasło: szeregi potęgowe; promień zbieżności

P $a_n = 1$; $\quad F_a(x) = \sum_{n \geq 0} 1 \cdot x^n = \sum_{n \geq 0} x^n = \frac{1}{1-x}$

$|x| < 1$

P Fix n;

$a_k = \binom{n}{k} \quad k \in \mathbb{N}$

$$F_a(x) = \sum_{k \geq 0} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n .$$

P

Liczby Fibonacciego :

$$F_0 = 0$$
$$F_1 = 1$$
$$F_{n+2} = F_n + F_{n+1} .$$

$$F(x) = \sum_{n \geq 0} F_n \cdot x^n =$$

$$= 0 \cdot x^0 + 1 \cdot x^1 + \sum_{n \geq 0} F_{n+2} \cdot x^{n+2} =$$

$$= x + \sum_{n \geq 0} (F_n + F_{n+1}) x^{n+2} = x + \sum_{n \geq 0} F_n x^{n+2} + \sum_{n \geq 0} F_{n+1} x^{n+2}$$

$$= x + x^2 \sum_{n \geq 0} F_n x^n + x \sum_{n \geq 0} F_{n+1} x^{n+1} = x + x^2 F(x) + x \sum_{n \geq 0} F_n x^n$$

$$= x + x^2 F(x) + x F(x)$$

$$F(x) = x + x \cdot F(x) + x^2 F(x)$$

$$F(x)\left(1 - x - x^2\right) = x$$

$$F(x) = \frac{x}{1 - x - x^2}$$

$w(x) = x^2 + x - 1$

$\Delta = 1 - 4(-1) = 5$

$\sqrt{\Delta} = \sqrt{5}$

$$f(x,y) = \frac{1}{1-(1+x)y}$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$

$$= \sum_{n\geq 0} \underbrace{\frac{f^{(n)}(0)}{n!}}_{a_n} x^n = \sum_{n\geq 0} a_n x^n .$$

$$f(x,y) = \sum_{n\geq 0} \frac{1}{n!}\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right\}^n (f)(0) .$$