LICZBY FIBONACCIEGO

$F_{n+2} = F_n + F_{n+1}$

$$F(x) = \sum_{n\geq 0} F_n \cdot x^n$$

$$F(x) = \frac{x}{1 - x - x^2}$$ ← funkcja wymierna

$x^2 + x - 1 = 0$ $\Delta = 1 + 4 = 5$

$\omega_1 = \frac{-1+\sqrt{5}}{2}$ $\omega_2 = \frac{-1-\sqrt{5}}{2}$

$$\begin{cases} \omega_1 \cdot \omega_2 = -1 \\ \omega_1 + \omega_2 = -1 \\ \omega_1 - \omega_2 = \sqrt{5} \end{cases}$$

$\frac{1}{\omega_1} = -\omega_2$

To nie przejdzie:

$F_{n+2} = F_n + n \cdot F_{n+1}$

$$F(x) = \frac{x}{1-x-x^2} = \frac{-x}{(x-\omega_1)(x-\omega_2)} =$$

$$= -\left(\frac{A}{x-\omega_1} + \frac{B}{x-\omega_2}\right) = \frac{A}{\omega_1 - x} + \frac{B}{\omega_2 - x}$$

$$= \frac{A(\omega_2 - x) + B(\omega_1 - x)}{(\omega_1 - x)(\omega_2 - x)} = \frac{-(A\omega_2 + B\omega_1) + (A+B)\cdot x}{-m(x)}$$

$$\begin{cases} A + B = 1 \\ A\omega_2 + B\omega_1 = 0 \end{cases}$$

$$\begin{cases} B = 1 - A \\ A\omega_2 + (1-A)\omega_1 = 0 \\ A(\omega_2 - \omega_1) + \omega_1 = 0 \end{cases}$$

$$A = \frac{-\omega_1}{\omega_2 - \omega_1} = \frac{\omega_1}{\omega_1 - \omega_2}$$

$$B = 1 - A = 1 - \frac{\omega_1}{\omega_1 - \omega_2} = \frac{\cancel{\omega_1} - \omega_2 - \cancel{\omega_1}}{\omega_1 - \omega_2} = \frac{\omega_2}{\omega_2 - \omega_1}$$

$$A = \frac{\omega_1}{\sqrt{5}} \qquad B = \frac{-\omega_2}{\sqrt{5}}$$

$$F(x) = \sum_n F_n x^n$$

$$F(x) = \frac{A}{\omega_1 - x} + \frac{B}{\omega_2 - x} = \frac{1}{\sqrt{5}}\left(\frac{\omega_1}{\omega_1 - x} - \frac{\omega_2}{\omega_2 - x}\right) = $$

$$= \frac{1}{\sqrt{5}}\left(\frac{1}{1-(x/\omega_1)} - \frac{1}{1-(x/\omega_2)}\right)$$

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n$$

$$|x| < 1$$

$$= \frac{1}{\sqrt{5}}\left(\sum_{n\geq 0}\left(\frac{x}{\omega_1}\right)^n - \sum_{n\geq 0}\left(\frac{x}{\omega_2}\right)^n\right)$$

$$= \frac{1}{\sqrt{5}}\sum_{n\geq 0}\left(\left(\frac{1}{\omega_1}\right)^n - \left(\frac{1}{\omega_2}\right)^n\right)x^n = \frac{1}{\sqrt{5}}\sum_{n\geq 0}\left((-\omega_2)^n - (-\omega_1)^n\right)x^n =$$

$$= \sum_{n\geq 0} \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(-\frac{\sqrt{5}-1}{2}\right)^n\right) x^n$$

$$= \sum_{n\geq 0} \underbrace{\frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - (-1)^n\left(\frac{\sqrt{5}-1}{2}\right)^n\right)}_{F_n} x^n$$

~~UWAGA: dzisiaj był błąd; zamiana~~ ~~ω_1 z ω_2~~

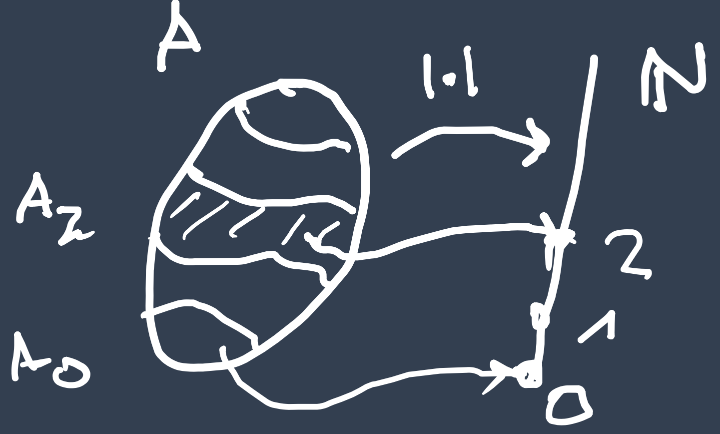
Uwaga: na wykładzie zapomniałem jednego minusa we wzorach na pierwiastki. Teraz jest ok.

KLASY KOMBINATORYCZNE

Def. Klasa kombin. nazywamy parę $\mathcal{A} = (A, |\cdot|)$ t.że $|\cdot| : A \longrightarrow \mathbb{N}$ oraz

$$(\forall n)\left(\left|\{a \in A : |a| = n\}\right| < \aleph_0\right).$$

Intuicja: $|a|$ = "rozmiar a"



- $A_n = \{a \in A : |a| = n\}$
- $a_n = |A_n|$

Wniosek: $|A| \leq \aleph_0$

Ustalmy klasę komb. $\mathcal{A} = (A, |\cdot|)$:

$$\cdot\ \mathcal{A}(x) = \sum_{n \geq 0} a_n x^n \leftarrow \text{funkcja tworząca } \mathcal{A}.$$

Oznaczenie: $[x^n]\left(\sum_{k \geq 0} a_k x^k\right) = a_n.$

(P) $\mathcal{N} = (\mathbb{N}, |\cdot|)$, $|n| = n$; (czyli $|\cdot| = id_{\mathbb{N}}$)

$$\mathcal{N}(x) = \sum_{n \geq 0} 1 \cdot x^n \underset{|x|<1}{=} \frac{1}{1-x}.$$

P

$$\mathcal{A} = \{ \underset{0}{\underset{\uparrow}{\bigcirc}} , \underset{1}{\underset{\uparrow}{\square}} , \underset{2}{\underset{\uparrow}{\triangle}} \}$$

$|\bigcirc| = 0$

$|\square| = 1$

$|\triangle| = 2$

$$A(x) = \sum_{n \geq 0} a_n x^n = 1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 =$$

$$= 1 + x + x^2$$

P

$$\mathcal{A} = \{ \underset{0}{\underset{\uparrow}{\varepsilon}} , \underset{1}{\underset{\uparrow}{\square}} , \underset{1}{\underset{\uparrow}{\boxed{}}} , \underset{3}{\underset{\uparrow}{\bigcirc}} , \underset{4}{\underset{\uparrow}{\triangle}} \}$$

$$A(x) = 1 + 2 \cdot x + 0 \cdot x^2 + 1 \cdot x^3 + 1 \cdot x^4 =$$

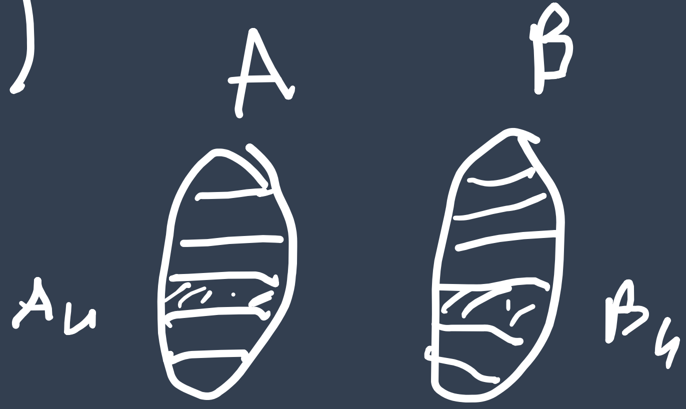
$$= 1 + 2x + x^3 + x^4 .$$

SUMA.

$$\mathcal{A} = (A, |\cdot|_1), \quad \mathcal{B} = (B, |\cdot|_2)$$

$$A \cap B = \emptyset$$

$$\mathcal{C} = \mathcal{A} + \mathcal{B} = (\underbrace{A \cup B}_{C}, \underbrace{|\cdot|_1 \cup |\cdot|_2}_{|\cdot|})$$



$$|c| = \begin{cases} |c|_1 : c \in A \\ |c|_2 : c \in B \end{cases}$$

$$C_n = \{c \in C : |c| = n\} = A_n \cup B_n$$

$$c_n = a_n + b_n$$

$$C(x) = \sum_{n\geq 0} c_n x^n = \sum_{n\geq 0} (a_n+b_n) x^n$$

$$= \sum_{n\geq 0} a_n x^n + \sum_{n\geq 0} b_n x^n = A(x) + B(x)$$

$$(A+B)(x) = A(x) + B(x)$$

$$A \cap B = \emptyset$$

ILOCZYN: mamy $\mathcal{A} = (A, |\cdot|_1)$, $\mathcal{B} = (B, |\cdot|_2)$.

$$\left.\begin{array}{l} C = A \times B \\ |(a,b)| = |a|_1 + |b|_2 . \end{array}\right\}$$

$(C, |\cdot|) \leftarrow$ kl. komb.

$\Uparrow$

$\mathcal{A} \times \mathcal{B}$

$$
\begin{aligned}
C_n &= \{(a,b) \in A \times B : |(a,b)| = n\} = \\
&= \{(a,b) \in A \times B : |a|_1 + |b|_2 = n\} \\
&= \bigcup_{k=0}^{n} \{(a,b) \in A \times B : |a|_1 = k \wedge |b|_2 = n-k\} \\
&= \dot{\bigcup_{k=0}^{n}} (A_k \times B_{n-k})
\end{aligned}
$$

$(\cdot =$ rozłączność$)$

skończ.

$$\mathcal{A} \times \mathcal{B} = (\underbrace{A \times B}_{C}, |\cdot|)$$

$$c_n = |C_n| = \left|\bigcup_{k=0}^{n} (A_k \times B_{n-k})\right| =$$

$$= \sum_{k=0}^{n} |A_k \times B_{n-k}| = \sum_{k=0}^{n} a_k \cdot b_{n-k}$$

$$(\mathcal{A} \times \mathcal{B})(x) = \sum_n c_n x^n = \sum_n \left(\sum_{k=0}^{n} a_k \cdot b_{n-k}\right) x^n$$

$$= \left(\sum_n a_n x^n\right) \cdot \left(\sum_n b_n x^n\right)$$

$$\boxed{(\mathcal{A} \times \mathcal{B})(x) = \mathcal{A}(x) \cdot \mathcal{B}(x)}$$

(P) $\mathcal{N} = (\mathbb{N}, |\cdot|)$, $|\cdot| = \mathrm{id}$.

$$\mathcal{N}(x) = \sum_n x^n = \frac{1}{1-x}$$

$$\mathcal{N} \times \mathcal{N} = (\mathbb{N} \times \mathbb{N}, |\cdot|) \qquad |(k, l)| = |k| + |l| = k + l$$

$$(\mathcal{N} \times \mathcal{N})(x) = \mathcal{N}(x) \cdot \mathcal{N}(x) =$$

$$= \left(\frac{1}{1-x}\right)^2 = (1-x)^{-2} = \sum_{n \geq 0} \binom{-2}{n} (-x)^n =$$

$$= \sum_{n \geq 0} \binom{-2}{n} (-1)^n x^n = \sum_{n \geq 0} \binom{n+2-1}{n} x^n$$

$$= \sum_{n \geq 0} \binom{n+1}{n} x^n = \sum_{n \geq 0} \binom{n+1}{1} x^n$$

$$= \sum_{n \geq 0} (n+1) x^n .$$

$$(N \times N)_n = \underbrace{\{(0,n), (1,n-1), \dots, (n,0)\}}_{n+1.}$$

$$|(N \times N)_n| = n+1 .$$

ZADANIE : $[x^n](N \times N \times N)(x) = ?$

CIĄGI

$\mathcal{A} = (A, |\cdot|)$

$$Seq(\mathcal{A}) = \mathcal{E} + \mathcal{A} + (\mathcal{A}\times\mathcal{A}) + (\mathcal{A}\times\mathcal{A}\times\mathcal{A}) + \ldots$$

$$\mathcal{E} = (\{\varepsilon\}, |\cdot|) \quad |\varepsilon| = 0 \qquad \mathcal{E}(x) = 1$$

Q: czy $Seq(\mathcal{A})$ jest klasą kombin.?

Zał. że jest $a \in A$ t. że $|a| = 0$.

$$|(a, a, a)| = |a| + |a| + |a| = 0$$

W $Seq(\mathcal{A})$ mamy ∞ wiele elem. rozmiaru 0.

FAKT: jeśli $a_0 = 0$, to $Seq(\mathcal{A})$ jest klasą kombin.

$$\{(a_1, \dots, a_k) \in A^* : |(a_1 \dots a_k)| = n\} =$$

$$= \{(a_1 \dots a_k) \in A^* : |a_1| + \dots + |a_k| = n\} \subseteq$$

$$n = |a_1| + \dots + |a_k| \geq 1 + 1 + \dots + 1 = k$$

$$\subseteq \{\emptyset\} \cup A \cup (A \times A) \cup \dots \cup \underbrace{(A \times \dots \times A)}_{n}$$

skończ.

$$\mathrm{Seq}(\mathcal{A}) = \mathcal{E} + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) + \dots$$

$$\mathrm{Seq}(\mathcal{A})(x) = \mathcal{E}(x) + \mathcal{A}(x) + (\mathcal{A} \times \mathcal{A})(x) + \dots$$

$$= 1 + \mathcal{A}(x) + \mathcal{A}(x)^2 + \mathcal{A}(x)^3 + \dots$$

$$= \frac{1}{1 - \mathcal{A}(x)}$$

$$\mathrm{Seq}(\mathcal{A})(x) = \frac{1}{1 - \mathcal{A}(x)}$$

$$\mathrm{Seq}(\mathcal{A})(x) = \frac{1}{1 - \mathcal{A}(x)}$$

P $\quad \mathcal{A} = \{\uparrow, \downarrow\} \qquad |\uparrow| = |\downarrow| = 1$

$$\mathcal{A}(x) = 2x$$

$\uparrow\downarrow\uparrow$

$\uparrow\uparrow\uparrow\uparrow$

$\uparrow\downarrow\downarrow\uparrow\uparrow$

$$\mathrm{Seq}(\mathcal{A})(x) = \frac{1}{1 - \mathcal{A}(x)} = \frac{1}{1 - 2x}$$

$$= \sum_{n \geq 0} (2x)^n = \sum_{n \geq 0} 2^n x^n$$

$$\boxed{[x^n]\, \mathrm{Seq}(\mathcal{A})(x) = 2^n}$$

P

$$\mathcal{D} = \Big\{ \underset{1}{\boxed{\cdot}}\ ,\ \underset{2}{\boxed{\cdot \,\big|\, \cdot}} \Big\}$$

$$\mathcal{D}(x) = 1 \cdot x^1 + 1 \cdot x^2 = x + x^2$$

$$\mathrm{Seq}(\mathcal{D})(x) = \frac{1}{1 - \mathcal{D}(x)} = \frac{1}{1 - x - x^2}$$

$$F(x) = \frac{x}{1 - x - x^2}$$

CO TU JEST GRANE ?