LICZBY CATALANA

① Ustalmy $n \in \mathbb{N}$. Rozważamy $n+1$ zmiennych $x_0, x_1, \ldots, x_n$. Ustalamy dział. binarne $*$. Ile różnych wyrażeń możemy zbudować?

$n=2$: $(x_0 * x_1) * x_2$ $\quad$ $x_0 * (x_1 * x_2)$ $\quad$ $L_2 = 2$

$n=3$: $x_0 * (x_1 * (x_2 * x_3))$ $\quad$ $(x_0 * x_1) * (x_2 * x_3)$

$((x_0 * x_1) * x_2) * x_3$ $\quad$ $(x_0 * (x_1 * x_2)) * x_3$

$x_0 * ((x_1 * x_2) * x_3)$ $\quad$ $L_3 = 5$

$n=1$: $x_0 * x_1$; $L_1 = 1$

$n=0$: x_0 ; $L_0 = 1$

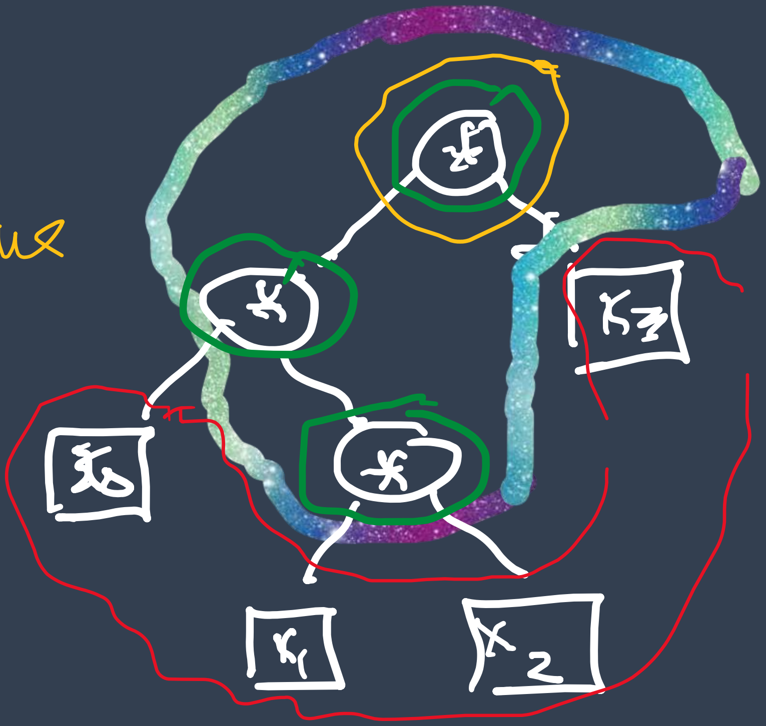
Inna reprezentacja

$n = 3$

$$(x_0 * x_1) * (x_2 * x_3)$$

$$(x_0 * (x_1 * x_2)) * x_3$$

węzły wewnętrzne



korzeń

liście

drzewa

liść ≡ nie ma potomka

$n = 0$ $\quad x_0 \quad$ $\boxed{x_0}$ ← liść $\quad$ wewn. = 0

$n = 1$ $\quad x_0 * x_1 \quad$ wewn = 1

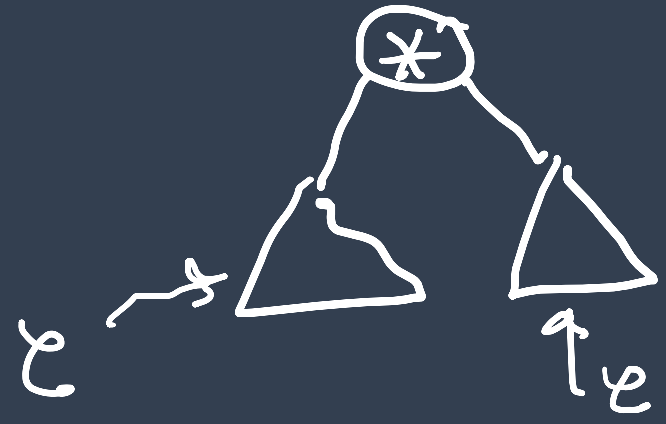
Równie dla klasy drzewbin.

$$\mathcal{C} = (M, |\cdot|)$$

M – napisy

$|m|$ = liczba *

$$\mathcal{C} \cong \underbrace{\{\varepsilon\}}_{0} + \mathcal{C} \times \underbrace{\{*\}}_{1} \times \mathcal{C}$$



$$\mathcal{C} = \{\varepsilon\} + \mathcal{C} \times \{*\} \times \mathcal{C}$$

$$\begin{aligned}
\mathcal{C}(x) &= \left(\{\varepsilon\} + \mathcal{C} \times \{*\} \times \mathcal{C}\right)(x) \\
&= \{\varepsilon\}(x) + \mathcal{C}(x) \cdot \{*\}(x) \cdot \mathcal{C}(x) \\
&= 1 + \mathcal{C}(x) \cdot x \cdot \mathcal{C}(x) \\
&= 1 + x \cdot \mathcal{C}^2(x).
\end{aligned}$$

$$x\,\mathcal{C}^2(x) - \mathcal{C}(x) + 1 = 0$$

$$x\, C^2(x) - C(x) + 1 = 0$$

$$x \cdot C^2 - C + 1 = 0$$

$$\Delta = 1 - 4x$$

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$\left\{ Q: + \text{ czy } - \right\}$

- rozw. $+$: podst. 0:

$$\frac{1+\sqrt{1-0}}{0} = \frac{2}{0} = \infty$$

- rozw. $-$: wst. 0;

$$\frac{1-\sqrt{1-0}}{0} = \frac{0}{0}$$

$$C(x) = \sum_{n \geq 0} C_n x^n$$

$$C(0) = C_0 = 1$$

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

$$= \frac{1}{2x}\left(1 - (1-4x)^{1/2}\right) = \frac{1}{2x}\left(1 - \sum_{n\geq 0} \binom{1/2}{n} (-4x)^n\right)$$

$$= \frac{(-1)}{2x} \sum_{n\geq 1} \binom{1/2}{n} (-1)^n 4^n x^n \qquad = -\frac{1}{2} \sum_{n\geq 1} \binom{1/2}{n} (-1)^n 4^n x^{n-1}$$

ZADANIE $\binom{1/2}{n} = (-1)^{n+1} \frac{2}{4^n} \frac{1}{n} \binom{2(n-1)}{n-1}$ $= \sum_{n\geq 1} \frac{1}{n} \binom{2(n-1)}{n-1} x^{n-1}$

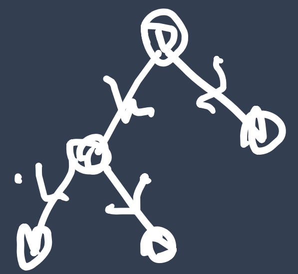
$$C(x) = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n$$

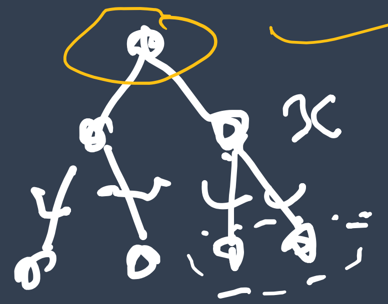
$$C_n = \frac{1}{n+1} \binom{2n}{n}$$: n-ta liczba Catalana

$$C_3 = \frac{1}{4} \binom{6}{3} = \frac{1}{4} \cdot \frac{6 \cdot 5 \cdot 4}{6} = 5.$$

Asymptotyka: $$C_n \sim \frac{4^n}{\sqrt{\pi}\, n^{3/2}}$$

Drzewa binarne





- korzeń
- liczba dzieci: 0 lub 2

- (ind. mat.) T – drz. binarne, n – wierz. wewn. → n+1 liści

ZADANIE

Wn. drzew binarnych o n-wierz. wewn. mamy C_n

T_n = liczba drzew o n-wierzch.

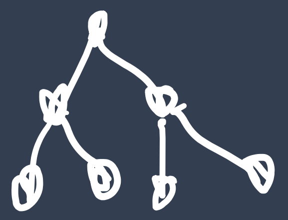
$$T_n = \begin{cases} 0 & : 2 \mid n \\ C_{\frac{n-1}{2}} & : \neg(2 \mid n) \end{cases}.$$

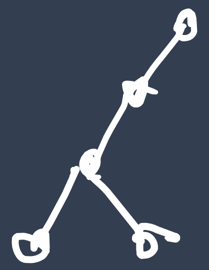
$n = 2m + 1$

$\frac{n-1}{2} = m$

Dowolne drzewa

korzeń





Równanie !

-  ciąg $\neq \emptyset$ drzew

$$Seq(\mathcal{A}) = \{\varepsilon\} + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) + \dots$$

$$Seq_{\neq 0}(\mathcal{A}) = \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + \dots$$

$$Seq_{\neq 0}(\mathcal{A})(x) = \mathcal{A}(x) + \mathcal{A}^2(x) + \mathcal{A}^3(x) + \dots$$

$$= \mathcal{A}(x)\left(1 + \mathcal{A}(x) + \mathcal{A}^2(x) + \dots\right) = \frac{\mathcal{A}(x)}{1 - \mathcal{A}(x)}$$

$$\mathcal{T} = \underset{\substack{\downarrow \\ 1}}{\{\bullet\}} + \underset{\substack{\downarrow \\ 1}}{\{\bullet\}} \times \mathrm{Seq}_{\neq 0}(\mathcal{T})$$

$$T(x) = x + x \cdot \frac{T(x)}{1 - T(x)} = x\left(1 + \frac{T(x)}{1-T(x)}\right)$$

$$= x\left(\frac{1 - T(x) + T(x)}{1 - T(x)}\right)$$

$$T(x) = \frac{x}{1 - T(x)}$$

$$\Delta = 1 - 4x$$

$$T = \frac{1 - \sqrt{1-4x}}{2}$$

STANDARDOWA metoda:

$$T = \frac{x}{1-T}$$

$$T(1-T) = x$$

$$T - T^2 = x$$

$$T^2 - T + x = 0$$

LAGRANGE INVERSION THEOREM !!!

Zał. że φ jest analityczna w 0

(tzn. rozwija się w szereg potęgowy o promieniu zbieżn. >0 w punkcie 0) i $\varphi(0) \neq 0$.

Istnieje wtedy funkcja f analityczna w 0

t. że $f(x) = x \cdot \varphi(f(x))$.

dla małych x. Co więcej

$$[x^n] f(x) = \frac{1}{n} [u^{n-1}] (\varphi(u))^n.$$

$$T(x) = x \cdot \frac{1}{1-T(x)} \qquad \varphi(u) = \frac{1}{1-u} = (1-u)^{-1}$$

$$T(x) = x \cdot \varphi(T(x))$$

$\cdot\ \varphi(0) = 1 \neq 0$

$\cdot\ \varphi(u) = \sum_{n} u^n$

$R = 1$

$$[x^n]T(x) = \frac{1}{n}[u^{n-1}]\varphi(u)^n =$$

$$= \frac{1}{n}[u^{n-1}](1-u)^{-n} = \frac{1}{n}[u^{n-1}]\sum_{k\geq 0}\binom{-n}{k}(-u)^k =$$

$$= \frac{1}{n}[u^{n-1}]\sum_{k\geq 0}\binom{k+n-1}{k}u^k = \frac{1}{n}\binom{2(n-1)}{n-1} = C_{n-1}\ !$$

$$\begin{cases} T_0 = 0 \ ; \ n>0 \Rightarrow T_n = C_{n-1} \end{cases}$$

$\blacksquare$

"Wyrażenia" raz jeszcze:

$$C(x) = 1 + x \cdot C^2(x)$$

$$C = 1 + x \cdot C^2$$

$$C = D + 1$$

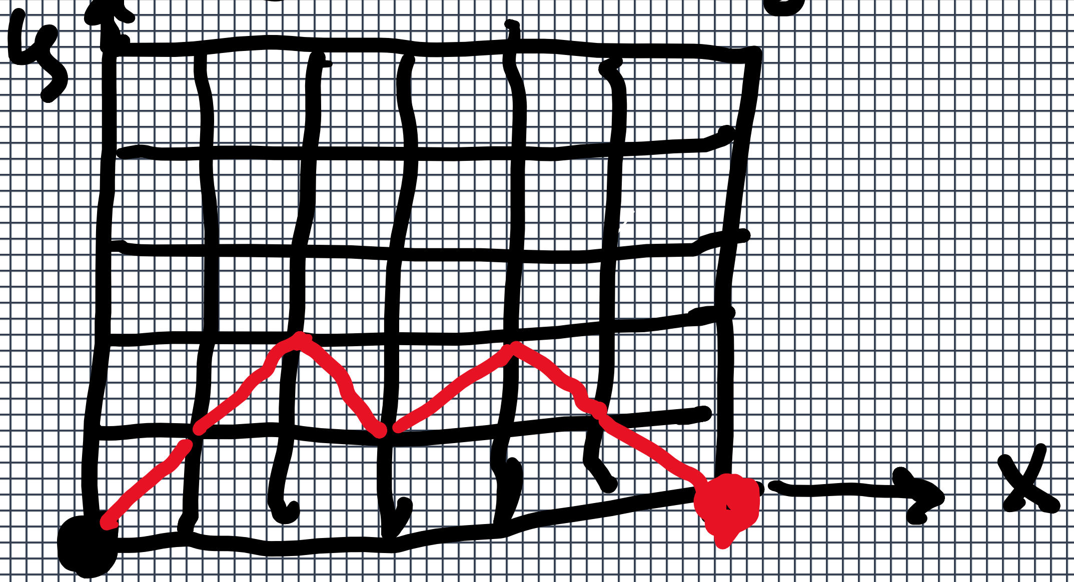
$$D + 1 = 1 + x(1+D)^2$$

$$D = x(1+D)^2 \qquad \varphi(u) = (1+u)^2$$

$$D_n = \frac{1}{n}[u^{n-1}](1+u)^{2n} = \frac{1}{n}[u^{n-1}]\sum_{k=0}^{2n}\binom{2n}{k}u^k = \ldots$$

$$= \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n}\frac{(2n)!}{(n-1)!(n+1)!} = \frac{1}{n+1}\binom{2n}{n} = C_n.$$

Ścieżki Dyke'a

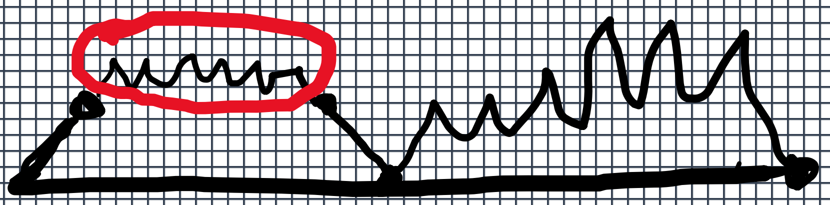


ruchy : $\nearrow$ $\searrow$

nie możemy zejść poniżej osi x.

START : $(0,0)$

STOP : $(2n,0)$



"równanie na D"

$$D = \{\varepsilon\} + (\nearrow \times D \times \searrow) \times D$$

$$D(x) = 1 + x^2 D^2(x)$$

zadania

$$D(x) = \sum D_{2n} x^{2n}$$

$$S(y) = \sum D_{2n} y^n$$

$$D(x) = S(x^2)$$

$$S(x^2) = 1 + x^2 \left(S(x^2)\right)^2$$

$$S(y) = 1 + y\, S(y)^2$$

$\vdots$

$$D_{2n} = C_n$$

zadanie: dokończ to.