UWAGI O PORZĄDKU LEKSYKOGRAFICZNYM

- $(X, \preceq)$ – liniowy porządek
- $\preceq_{lex}$ – porz. leksyk. na X^*
- Tw. $(X^*, \preceq_{lex})$ jest liniowym porządkiem

(P) $X \supseteq \{a, b\}$, $a \neq b$; $a \prec b$.

- $a \prec_l aa \prec_l aaa \prec_l \dots$ ← nieskończ. rosnący ciąg
- $a \prec_l aa \prec_l aaa \prec_l \dots \prec b \prec ba \prec baa \prec \dots$
- $b \succ_{lex} ab \succ aab \succ aaab \succ aaaab \succ \dots$ ← nieskończ. ciąg malejący

[illegible]:

- $\varepsilon \preceq_{l} x \qquad \forall x \in X^{*}$
- $l(x) \neq \emptyset \longrightarrow \varepsilon \prec_{l} x$

- $x = (a, x_1, \dots, x_n)$

  $y = (b, y_1, \dots, y_m)$

  - $a \prec b \longrightarrow x \prec_{lex} y$
  - $b \prec a \longrightarrow y \prec_{lex} x$
  - $a = b \longrightarrow (x \preceq_{lex} y) \equiv (x_1, \dots, x_n) \preceq_{lex} (y_1 \dots y_m)$

PSEUDO-KOD!

```
const GT = 1
const EQ = 0
const LT = -1

function CMP(x:string, y:string): int {
        if (x="" AND y="") {return EQ;}
        if (x="" AND y<>"") {return LT;}
        if (x≠"" AND y="") {return GT;}
        x = [a|x1];
        y = [b|y2];
        if (a<b) {return LT}
        if (b<a) {return GT}
        return CMP(x1,y1);
}
```

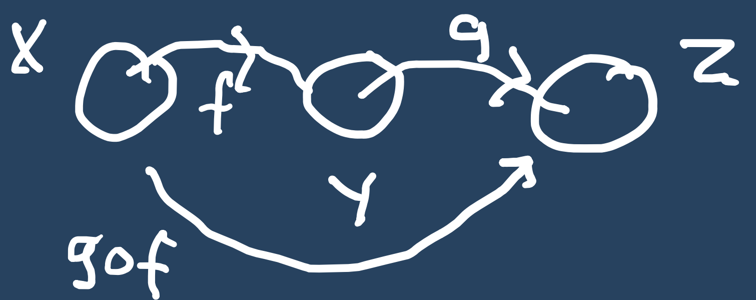
TEORIA MOCY

Def. $(|X| = |Y|) \equiv (\exists f)(f: X \xrightarrow[na]{1-1} Y)$

// X i Y są równoliczne

// alt. oznaczenie: $(\overline{X} = \overline{\overline{Y}})$

- $|X| = |X|$
- $|X| = |Y| \longrightarrow |Y| = |X|$
- $(|X| = |Y| \wedge |Y| = |Z|) \longrightarrow |X| = |Z|$

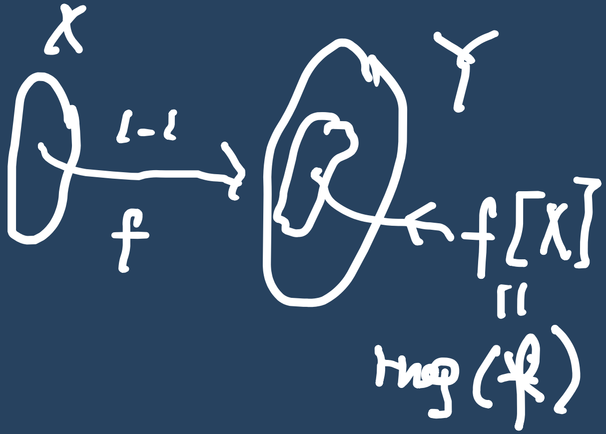


Def. $|X| \leq |Y| \equiv (\exists f)(f : X \xrightarrow{1-1} Y)$

Def. $|X| < |Y|$

$$\equiv$$

$$|X| \leq |Y| \wedge \neg(|X| = |Y|)$$



Tw (Cantor) $(\forall X)(|X| < |P(X)|)$

D-d. (1) Ustalmy X. Określamy $\varphi : X \to P(X)$ wzorem $\varphi(x) = \{x\}$. Wtedy φ jest 1-1.
Zatem $|X| \leq |P(X)|$.

(2) Zał., że $|X| = |P(X)|$. Mamy $f : X \xrightarrow[1-1]{na} P(X)$.
Niech $A = \{x \in X : x \notin f(x)\}$.
Wtedy $A \subseteq X$. Jest więc $a \in X$ t.że $A = f(a)$.
Wtedy:
$$a \in f(a) \leftrightarrow a \in A \leftrightarrow a \notin f(a).$$
Sprzeczność $\boxtimes$

P $|\mathbb{N}| < |P(\mathbb{N})| < |P(P(\mathbb{N}))| < \ldots$

Istnieje ∞ wiele ~~∞~~ różnych ∞.

UWAGA: jeśli zastosujemy tw Cantora do zbioru $X = \{1, \ldots, n\}$ $(n \in \mathbb{N})$ to

$$n < |P(X)| = 2^n$$

czyli $(\forall n)(n < 2^n)$.

// zadanie: zrób to ind mat

(P) Pok. że $|\mathbb{N}| \neq |P(\mathbb{N})|$.

Metoda przekątniowa

Zał. że $|\mathbb{N}| = |P(\mathbb{N})|$.

$$f : \mathbb{N} \xrightarrow[\text{na}]{1-1} P(\mathbb{N})$$

Dla $i, j \in \mathbb{N}$: $a_{ij} = \begin{cases} 1 : j \in f(i) \\ 0 : j \notin f(i) \end{cases}$

$$\begin{array}{r|cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & & & & \\
\hline
f(0): & \boxed{1} & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & \dots \\
f(1) & 0 & \boxed{1} & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & \dots \\
f(2) & 1 & 1 & \boxed{0} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \dots \\
 & 1 & 0 & 1 & \boxed{0} & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\
 & \vdots & \vdots & \vdots & \vdots & & \ddots & & & & \\
b: & 0 & 0 & 1 & 1 & & & & & &
\end{array}$$

$$\begin{array}{llll}
a_{00} & a_{01} & a_{02} & \dots \\
a_{10} & a_{11} & a_{12} & \dots \\
a_{20} & a_{21} & a_{22} &
\end{array}$$

$b_i = 1 - a_{ii}$

$B = \{ i : b_i = 1 \}$

$B \neq f(0), f(1), f(2)$

$(\forall n)(B \neq f(n))$.

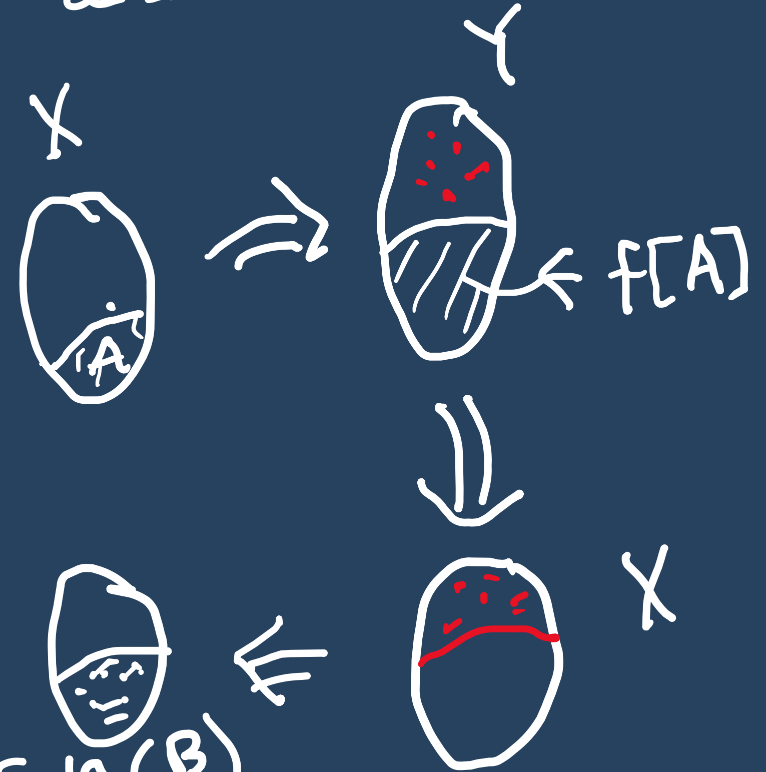
Tw (Cantor-Bernstein)

$$(|X| \leqslant |Y| \wedge |Y| \leqslant |X|) \to (|X| = |Y|)$$

<u>D-d</u>. Zał. że $f: X \xrightarrow{1-1} Y$ i $g: Y \xrightarrow{1-1} X$.

Definiujemy $\varphi: P(X) \to P(X)$ wzorem

$$\varphi(A) = X \setminus g\left[Y \setminus f[A]\right]$$



(F1) $A \subseteq B \subseteq X \to \varphi(A) \subseteq \varphi(B)$

D-d. $A \subseteq B \to f[A] \subseteq f[B] \to$

$\to Y \setminus f[A] \supseteq Y \setminus f[B] \to$

$\to g[Y \setminus f[A]] \supseteq g[Y \setminus f[B]]$

$\to X \setminus g[Y \setminus f[A]] \subseteq X \setminus g[Y \setminus f[B]] \leftrightarrow \varphi(A) \subseteq \varphi(B)$.

F2. Niech $(A_n)_{n\in\mathbb{N}}$ będzie ciągiem podzb. X. Wtedy

$$\boxed{\varphi\left(\bigcup_n A_n\right) = \bigcup_n \varphi(A_n).}$$

D-d. $\varphi\left(\bigcup_n A_n\right) = X \setminus g\left[Y \setminus f\left[\bigcup_n A_n\right]\right] =$

$= X \setminus g\left[Y \setminus \bigcup_n f[A_n]\right] \overset{\text{d.M.}}{=} X \setminus g\left[\bigcap_n (Y \setminus f[A_n])\right] \overset{g \text{ jest } 1-1}{=}$

$= X \setminus \bigcap_n g[Y \setminus f[A_n]] \underset{\text{d.M.}}{=} \bigcup_n \left(X \setminus g[Y \setminus f[A_n]]\right)$

$= \bigcup_n \varphi(A_n).$

$$\varphi(A) := X \setminus g[Y \setminus f[A]]$$

Definiujemy : $\begin{cases} N_0 = \emptyset \\ N_{n+1} = \varphi(N_n) \end{cases}$

czyli $\emptyset, \varphi(\emptyset), \varphi(\varphi(\emptyset)), \varphi(\varphi(\varphi(\emptyset)))$, ...

- $N_0 \subseteq N_1$ (bo $\emptyset \subseteq \varphi(\emptyset)$)
- $\varphi(N_0) \subseteq \varphi(N_1)$, czyli $N_1 \subseteq N_2$
- $\varphi(N_1) \subseteq \varphi(N_2)$, czyli $N_2 \subseteq N_3$
- (ind. po n pokaż, że $N_n \subseteq N_{n+1}$)

czyli

$$N_0 \subseteq N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$$

KŁADZIEMY : $N = \bigcup_n N_n$.

FAKT: $\boxed{\varphi(N) = N}$

D-d. $\varphi(N) = \varphi\left(\bigcup_n N_n\right) = \bigcup_n \varphi(N_n) =$

$$= \bigcup_{n \geq 0} (N_{n+1}) = \bigcup_{n \geq 1} N_n = \underset{\substack{\shortparallel \\ \emptyset}}{N_0} \cup \bigcup_{n \geq 1} N_n$$

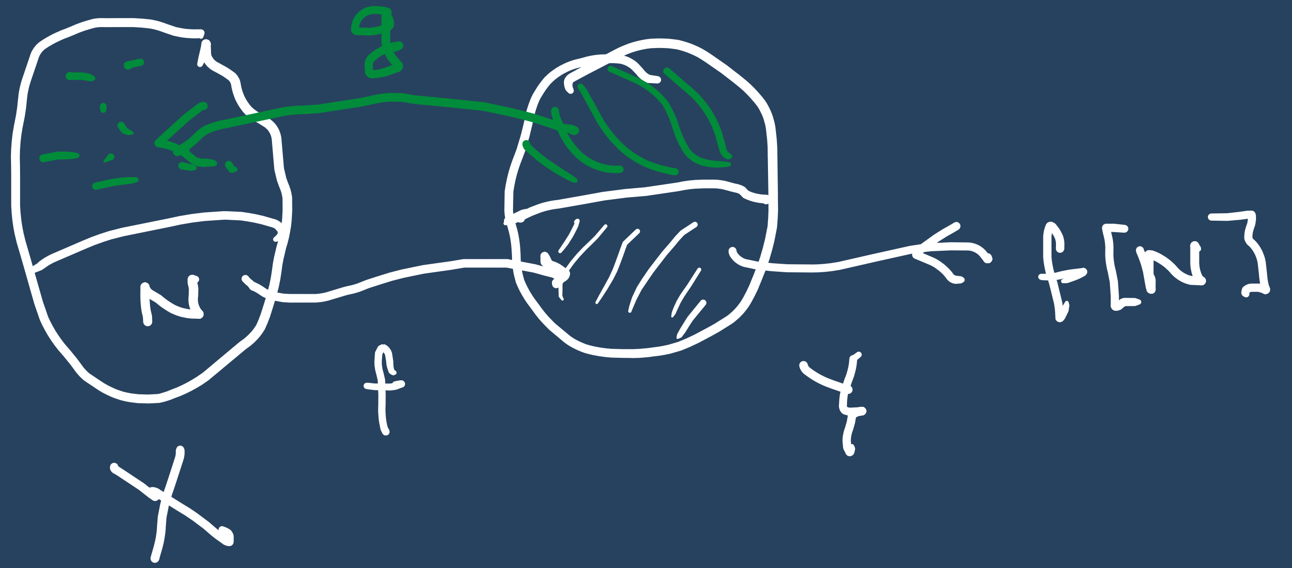
$$= \bigcup_{n \geq 0} N_n = N.$$

$\eta : Y \to Y$

$\eta(y) = y$

$y \leftarrow$ punkt stały η

fix point



$$N = X \setminus g[Y \setminus f[N]]$$

$$X \setminus N = g[Y \setminus f[N]]$$

$$h = (f \restriction N) \cup (g \restriction (Y \setminus f[N]))^{-1}$$

WTEDY: $h : X \xrightarrow[1-1]{na} Y$; CZYLI $|X| = |Y|$ $\blacksquare$