Przypomnienie:

Def. A - przelicz. $\equiv (A = \emptyset) \vee (\exists f)(f : \mathbb{N} \xrightarrow{na} A)$

Tw. A - przelicz $\equiv (\exists n \in \mathbb{N})(|A| = n) \vee$
$|A| = \aleph_0$.

FAKT:
- $|\mathbb{N} \times \mathbb{N}| = \aleph_0$
- $|\mathbb{Q}| = \aleph_0$

Tw. Zał. że $(A_n)_{n \in \mathbb{N}}$ jest rodziną zbiorów przeliczalnych (czyli: $(\forall n)(A_n$ jest przelicz$)$.) wtedy $\bigcup_{n \in \mathbb{N}} A_n$ jest przeliczalny.

D-d. Ustalmy rodz. $(A_n)_{n\in\mathbb{N}}$ zbiorów przelicz. Możemy zał. że $(\forall n)(A_n \neq \emptyset)$. (dlaczego?)

Dla każdego $n \in \mathbb{N}$ ustalmy

$$f_n : \mathbb{N} \xrightarrow{\text{"na"}} A_n .$$

// subtelność Aksjomat wyboru

Definiujemy $\varphi : \mathbb{N}\times\mathbb{N} \longrightarrow \bigcup_n A_n$:

$$\varphi((n,k)) = f_n(k).$$

Pok. że φ jest "na" $\bigcup_n A_n$.

Weźmy $a \in \bigcup_n A_n$. Jest $n\in\mathbb{N}$ t. że $a \in A_n$.

Jest $k \in \mathbb{N}$ t. że $f_n(k) = a$. Wtedy

$$\varphi((n,k)) = f_n(k) = a.$$

$$\begin{tikzcd} \mathbb{N}\times\mathbb{N} \arrow[r, "\varphi", "\underline{\text{na}}"'] & \bigcup_n A_n \\ \mathbb{N} \arrow[u, "\psi"', "\underline{\text{na}}\ 1\text{-}1"] \arrow[ur, "\text{“na”}", "\varphi\circ\psi"'] & \end{tikzcd}$$

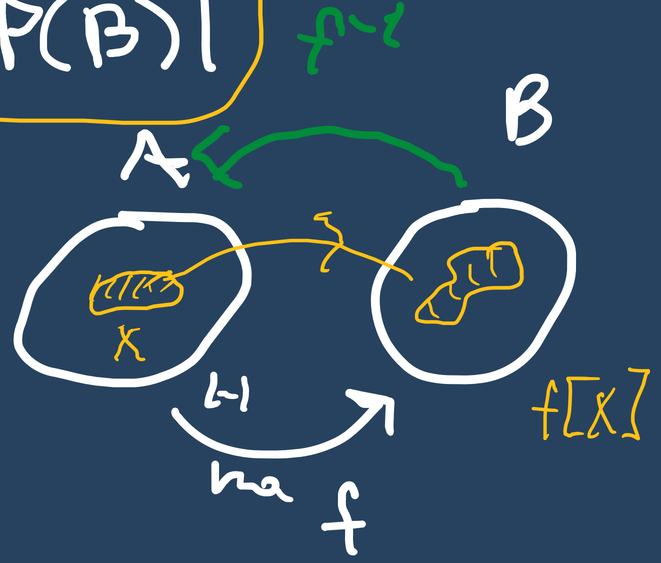
$\blacksquare$

Zbiory mocy kontinuum:

$$|A| = \mathfrak{c} \equiv |A| = |\mathbb{R}|.$$

FAKT 1. $|A| = |B| \longrightarrow |P(A)| = |P(B)|$

D-d. Ustalmy $f : A \xrightarrow[na]{1-1} B$.



Dla $X \in P(A)$ określamy

$$\varphi(X) = f[X].$$

- jasne: $\varphi : P(A) \longrightarrow P(B)$
- $\varphi(X) = \varphi(Y) \equiv f[X] = f[Y] \Longrightarrow f^{-1}[f[X]] = f^{-1}[f[Y]]$

$$\equiv (f^{-1} \circ f)[X] = (f^{-1} \circ f)[Y] = id_A[X] = id_A[Y]$$

$$\Rightarrow X = Y.$$

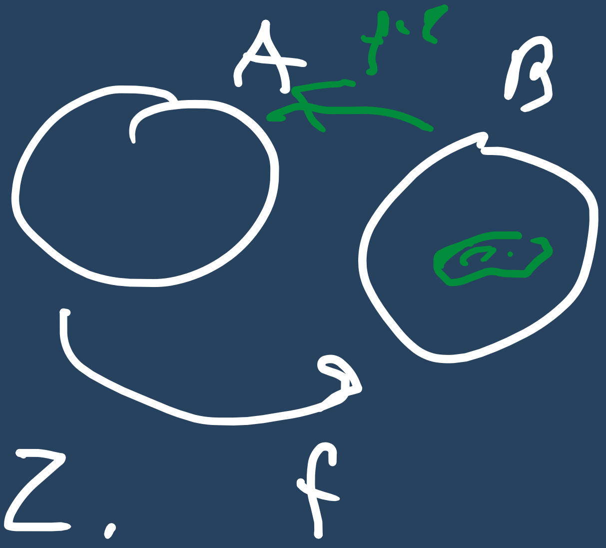
φ jest 1-1

- Bierzemy $Z \in P(B)$

Niech $X = f^{-1}[Z]$.

Wtedy $\varphi(X) = f[f^{-1}[Z]]$

$= (f \circ f^{-1})[Z] = id_B[Z] = Z.$



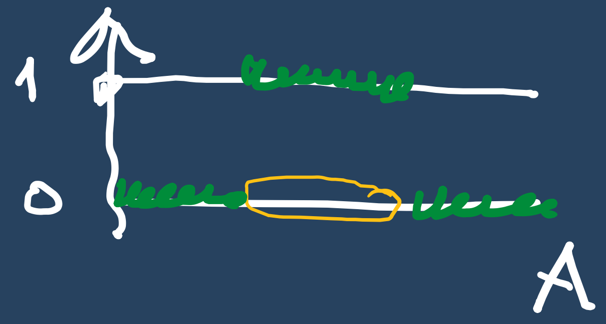
φ jest "na" $P(B)$ $\boxtimes$

Wniosek: $|P(\mathbb{Q})| = |P(\mathbb{N})|$

bo $|\mathbb{Q}| = |\mathbb{N}|$.

FAKT 2. $|P(A)| = |\{0,1\}^A|$

D-d. Dla $X \subseteq A$ określamy



$$1_X(a) = \begin{cases} 1 & : a \in X \\ 0 & : a \in A \setminus X \end{cases}$$

oraz $\varphi(X) = 1_X$ dla $X \subseteq A$.

- $\varphi : P(A) \longrightarrow \{0,1\}^A$
- zał. że $\varphi(X) = \varphi(Y)$; wtedy $1_X = 1_Y$.

zatem

$$(a \in X) \Longleftrightarrow (1_X(a) = 1) \Longleftrightarrow (1_Y(a) = 1) \Longleftrightarrow (a \in Y)$$

dla $a \in A$.

zatem $X = Y$.

• weźmy $h \in \{0,1\}^A$.

niech $X = h^{-1}[\{1\}]$,

$$(\varphi(X))(a) = 1 \equiv 1_X(a) = 1$$

$$\equiv a \in X \equiv a \in h^{-1}[\{1\}] \equiv h(a) = 1$$

zatem $\varphi(X) = h$.

A

$$\varphi(X) = 1_X$$

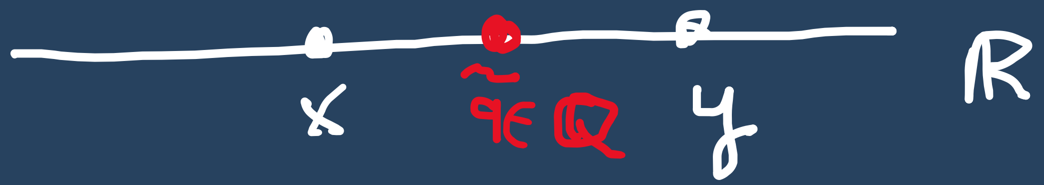
FAKT 3: $|\mathbb{R}| \leqslant |P(\mathbb{Q})| \quad (= P(\mathbb{N}))$

<u>D-d</u>. Dla $x \in \mathbb{R}$:

$$\varphi(x) = \{q \in \mathbb{Q} : q < x\}.$$

Wtedy $\varphi : \mathbb{R} \longrightarrow P(\mathbb{Q})$.

Weźmy $x, y \in \mathbb{R}$, $x \neq y$. Możemy zał. że $x < y$.



Niech $\tilde{q} \in \mathbb{Q}$ t. że $x < \tilde{q} < y$.

- $\tilde{q} \notin \varphi(x)$
- $\tilde{q} \in \varphi(y)$

zatem $\varphi(x) \neq \varphi(y)$

ZATEM φ jest 1-1. $\blacksquare$

FAKT 4. $|\{0,1\}^{\mathbb{N}}| \leq |\mathbb{R}|$

D-d. Określamy $\varphi : \{0,1\}^{\mathbb{N}} \longrightarrow \mathbb{R}$ wzorem

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{2x_n}{3^n}.$$

- φ jest dobrze określona

$$0 \leq \varphi(x) \leq \sum_{n=0}^{\infty} \frac{1}{3^n} \underset{\text{LO}}{=} \frac{1}{1-\frac{1}{3}} = \frac{3}{2} < \infty.$$

- Pok. że φ jest 1-1.

Weźmy $x, y \in \{0,1\}^{\mathbb{N}}$ t. że $x \neq y$.

Niech $k = \min\{n \in \mathbb{N} : x_n \neq y_n\}$

Możemy zał. że $x_k = 0$ i $y_k = 1$

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{x_n}{3^n} = \sum_{n<k} \frac{x_n}{3^n} + \underbrace{\frac{x_k}{3^k}}_{0} + \sum_{n=k+1}^{\infty} \frac{x_n}{3^n}$$

$$= \sum_{n<k} \frac{y_n}{3^n} + \sum_{n=k+1}^{\infty} \frac{x_n}{3^n} \leq \sum_{n<k} \frac{y_n}{3^n} + \sum_{n \geq k+1} \frac{1}{3^n}$$

$$= \sum_{n<k} \frac{y_n}{3^n} + \frac{1}{3^{k+1}} \sum_{n=0}^{\infty} \frac{1}{3^n} = \sum_{n<k} \frac{y_n}{3^n} + \frac{1}{3^{k+1}} \cdot \frac{3}{2}$$

$$= \sum_{n<k} \frac{y_n}{3^n} + \frac{1}{2} \cdot \frac{1}{3^k} < \sum_{n<k} \frac{y_n}{3^n} + \frac{\overset{=1}{y_k}}{3^k} + \sum_{n>k} \frac{0}{3^n}$$

$$\leq \sum_{n<k} \frac{y_n}{3^n} + \frac{y_k}{3^k} + \sum_{n>k} \frac{y_n}{3^n} = \varphi(y) \qquad \blacksquare$$

Tw. $|\mathbb{R}| = |P(\mathbb{N})|$

D-d.

$$|P(\mathbb{N})| = |\{0,1\}^{\mathbb{N}}| \leq |\mathbb{R}| \leq |P(\mathbb{Q})| = |P(\mathbb{N})|$$

$$|P(\mathbb{N})| \leq |\mathbb{R}| \leq |P(\mathbb{N})|$$

z tw. Cant.-Berns. mamy równość.

WNIOSEK: $|\mathbb{R}| > |\mathbb{N}|$

czyli: $\boxed{\aleph_0 < \mathfrak{c}}$

D-d: z tw. Cantora mamy

$|P(\mathbb{N})| > |\mathbb{N}|$

$\blacksquare$

ARYTMETYKA LICZB KARDYNALNYCH.

Jakie l. kard. znamy?

- n (dla $n \in \mathbb{N}$), $\aleph_0$, $\mathfrak{c}$
- $|P(\mathbb{R})| > |\mathbb{R}|$

<u>Def.</u> Niech κ, λ będą l. kard.

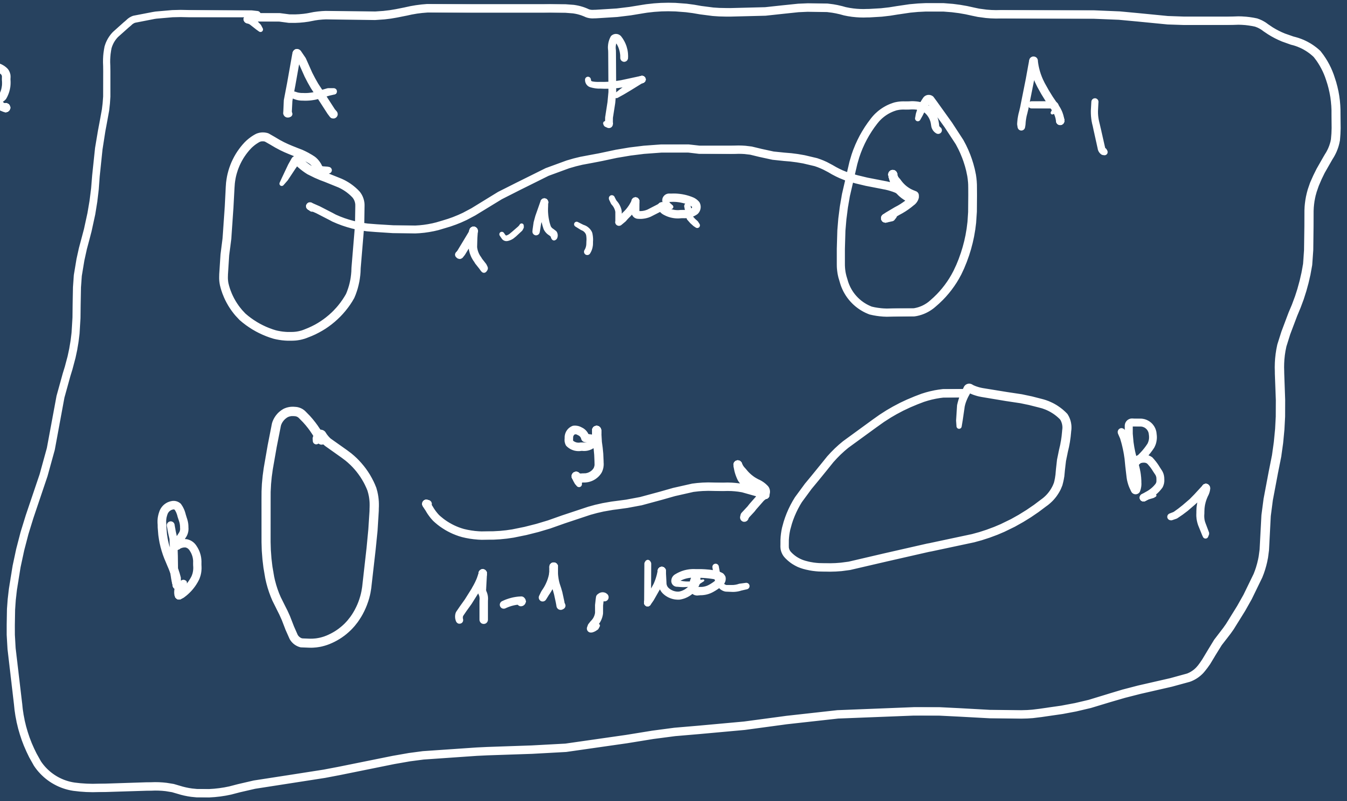
Niech $|A| = \kappa$, $|B| = \lambda$.

$$\kappa + \lambda = |(A \times \{0\}) \cup (B \times \{1\})|$$

Uwaga:
- $(A \times \{0\}) \cap (B \times \{1\}) = \emptyset$
- $|A \times \{0\}| = |A|$ $\quad$ $f(a) = (a, 0)$; $f: A \xrightarrow[\text{na}]{1-1} A \times \{0\}$

FAKT. $\left.\begin{array}{l} A \sim A_1 \\ B \sim B_1 \\ A \cap B = A_1 \cap B_1 = \emptyset \end{array}\right] \Longrightarrow |A \cup B| = |A_1 \cup B_1|$

D-d



niech

$h = f \cup g$

$h : A \cup B \to A_1 \cup B_1$

wtedy

h jest bijekcją

P1

$$1 + 2 = |(\{1\} \times \{0\}) \cup (\{1,2\} \times \{1\})|$$

$$= |\{1\} \cup \{2,3\}| = |\{1,2,3\}| = 3$$

$$1 + \aleph_0 = |(\{0\} \times \{0\}) \cup (\mathbb{N} \times \{1\})|$$

$$= |\{0\} \cup \mathbb{N}^+| = |\mathbb{N}|$$

$$1 + \aleph_0 = \aleph_0 \qquad \aleph_0 + \aleph_0 = \aleph_0$$

$$\aleph_0 + \aleph_0 = |(\mathbb{N} \times \{0\}) \cup (\mathbb{N} \times \{1\})| =$$

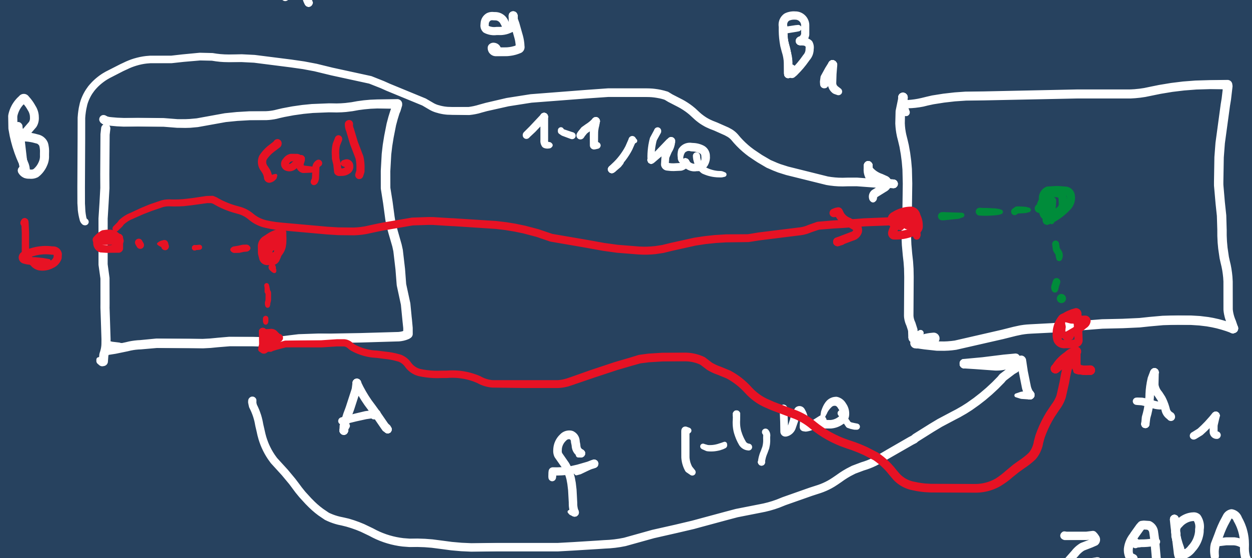
$$= |\mathbb{Z}^- \cup \mathbb{N}| = |\mathbb{Z}| = \aleph_0$$

Def. Niech $\kappa = |A|$, $\lambda = |B|$.

$$\kappa \cdot \lambda = |A \times B|.$$

Fakt. $\left.\begin{array}{l} A \sim A_1 \\ B \sim B_1 \end{array}\right] \Longrightarrow |A \times B| = |A_1 \times B_1|$

D-d.



$\varphi : A \times B \to A_1 \times B_1$

$\varphi((a,b)) = (f(a), g(b))$

ZADANIE: φ jest bij.

P $\quad 3 \cdot \aleph_0 = |\{1,2,3\} \times \mathbb{N}|$

$$\{1\} \times \mathbb{N} \subseteq \{1,2,3\} \times \mathbb{N} \subseteq \mathbb{N} \times \mathbb{N}$$

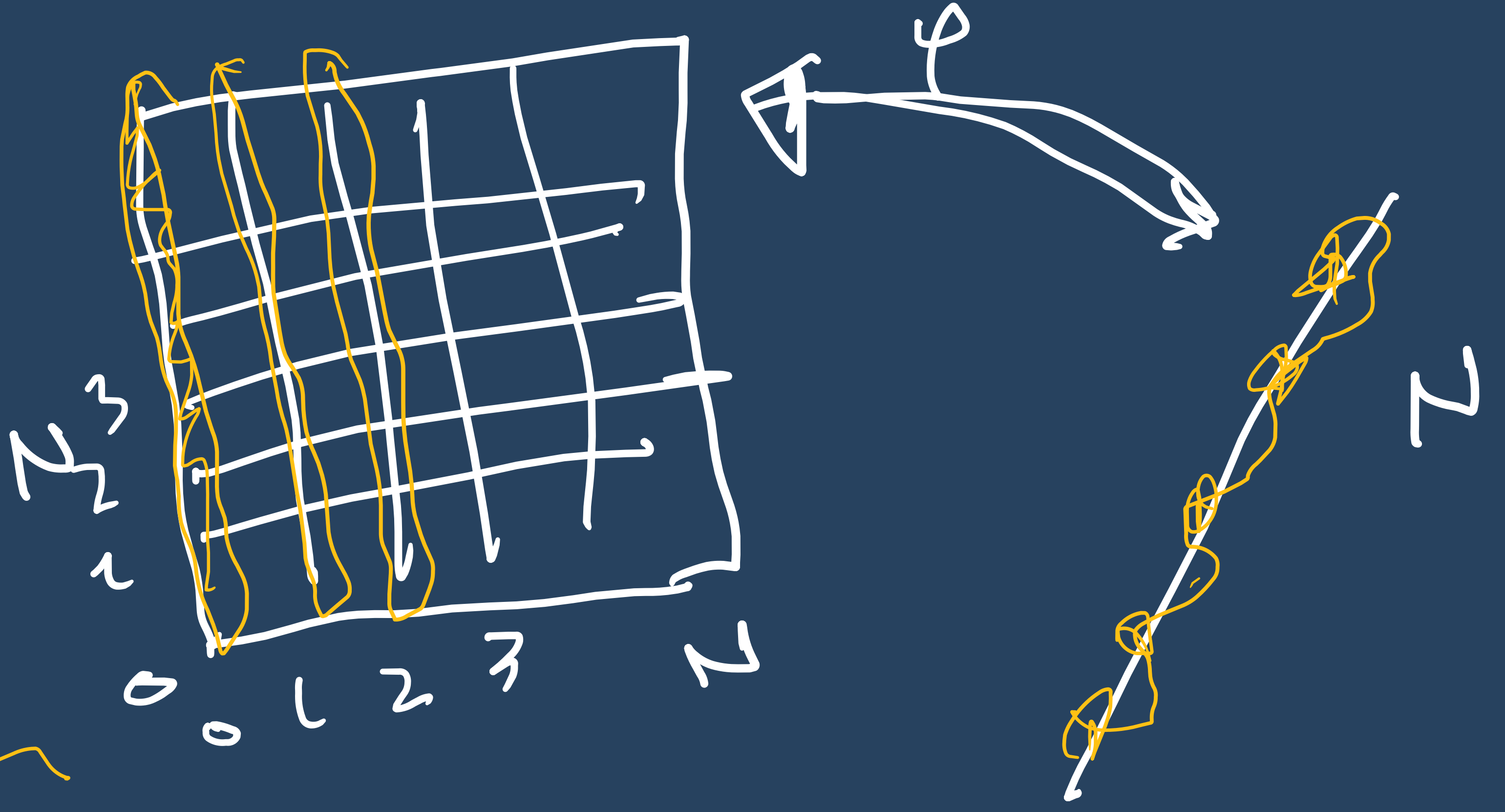
$$\aleph_0 = |\{1\} \times \mathbb{N}| \leq |\{1,2,3\} \times \mathbb{N}| \leq |\mathbb{N} \times \mathbb{N}| = \aleph_0$$

$$3 \cdot \aleph_0 = \aleph_0$$

P $\quad \aleph_0 \cdot \aleph_0 = \aleph_0.$

DYSKUSJE PO WYKŁADZIE :

$N \times N \sim N$



$$A_k = \varphi^{-1}\big(\overbrace{\{k\}\times N}^{N_2}\big)$$

$\{A_k\}_{k\in N} \leftarrow$ rozbicie N

na zbiory mocy $\aleph_0$

$$\varphi((n,k)) = 2^n(2k+1) - 1.$$

- weźmy $a \in \mathbb{N}$
- niech $b = a+1$.

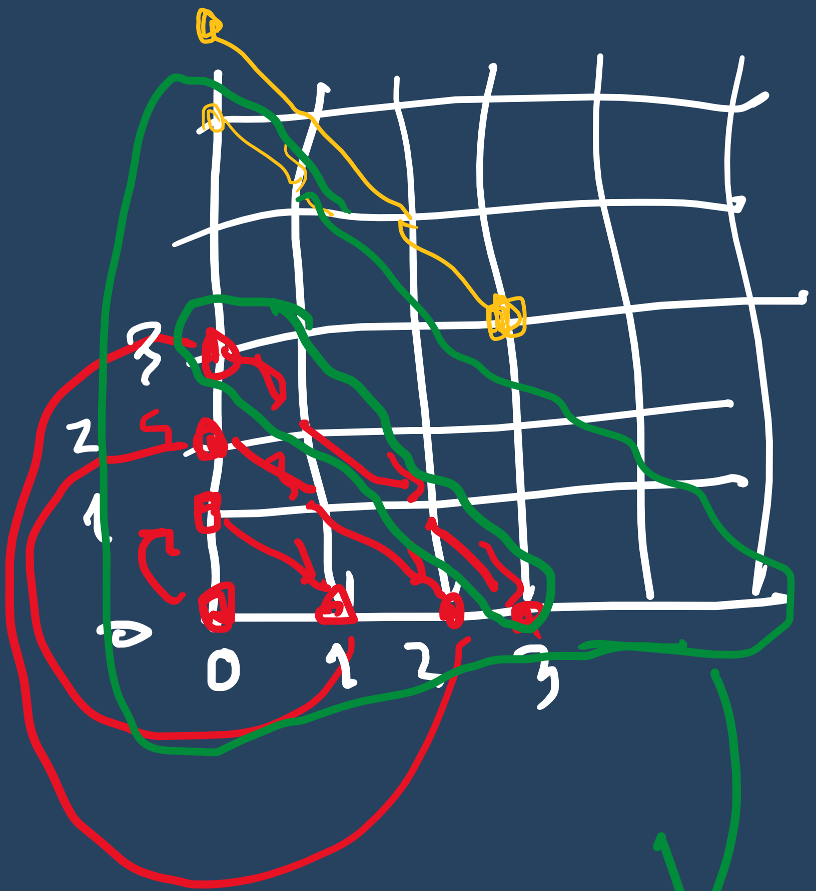
~~ktr~~ $b = 2^n(2k+1)$

- Znajdź najw n t.że

$$2^n \mid b$$

- $\frac{b}{2^n} = 2k+1$ ~~&~~ ; $k = \left(\frac{b}{2^n} - 1\right)\frac{1}{2}$

$$\varphi^{-1}(a) = (n,k)$$



$$P_k = \{(a,b) \in N^2 : a+b = k\}$$

$$|P_k| = k+1$$

$$|P_0 \cup P_1 \cup \dots \cup P_k| = 1 + 2 + \dots + (k+1) = \frac{(k+1)(k+2)}{2}$$

...

$\mathbb{N} = (2\mathbb{N}) \dot{\cup} (2\mathbb{N}+1)$

$\mathbb{N} = 3\mathbb{N} \dot{\cup} (3\mathbb{N}+1) \dot{\cup} (3\mathbb{N}+2)$

$\aleph_0 = \aleph_0 + \aleph_0$

$\aleph_0 = \aleph_0 + \aleph_0 + \aleph_0$.

$$\mathbb{N} = \bigcup_{n=0}^{\infty} \{n\} = \bigcup \underbrace{\{\{0\}, \{1\}, \{2\}, \{3\}, \dots\}}_{\aleph_0}$$

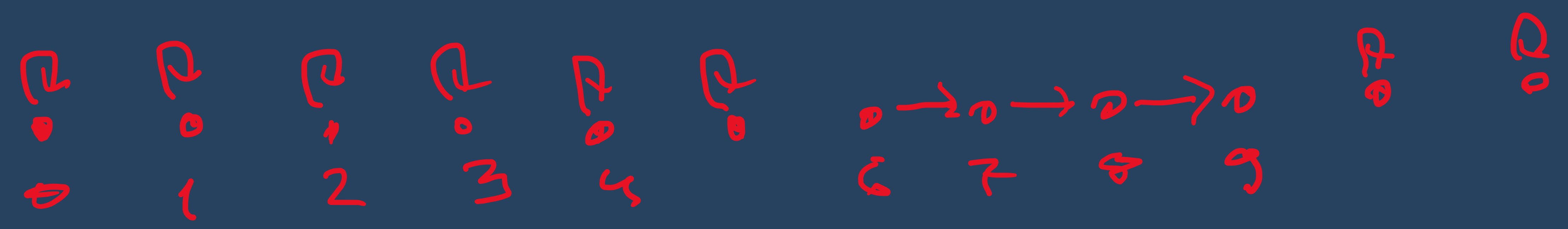
$\aleph_0 \cdot 1 = \aleph_0$

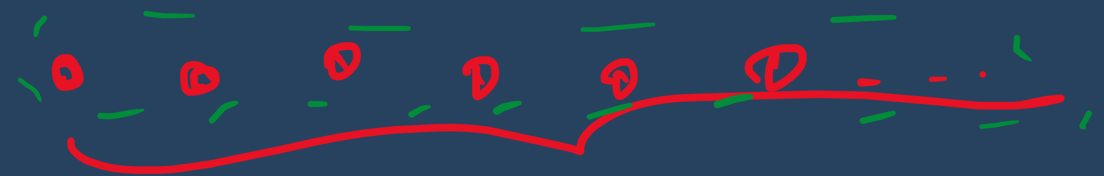
$\left.\begin{matrix} \kappa \le \lambda \\ \lambda \le \kappa \end{matrix}\right] \to \kappa = \lambda$

$$= \bigcup \{\{0,1\}, \{2,3\}, \{4,5\}, \{6,7\}, \dots\}$$

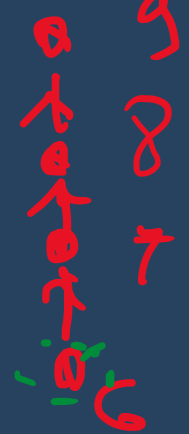
$\aleph_0 \cdot 2 = \aleph_0$.

$$\aleph_0 \aleph_0 = 1 \cdot \aleph_0 \overset{=}{\le} 2 \cdot \aleph_0 \overset{=}{\le} 3 \cdot \aleph_0 \overset{=}{\le} \dots \overset{=}{\le} \aleph_0 \cdot \aleph_0 = \aleph_0$$





$$\mathbb{N} \setminus \{6,7,8,9\}$$



$$\leq \restriction \{6, .. 9\}$$
$$\cup$$
$$\leq \restriction \{8, 11\}$$
$$\cup$$
$$id_{\mathbb{N}}$$

$$R = \left(\leq \restriction \{6, .. 9\} \right) \cup id_{\mathbb{N}}$$

$R \cup Q \ni (0,1), (1,0)$

$R = \{(0,0),(0,1),(1,1)\}$

$Q = \{(0,0),(1,1),(1,0)\}$

$|\{0,1\}^{A}| = 2^n$

$|P(\{1,\ldots,n\})| = 2^n$

dzisiaj $|\{0,1\}^A| = |P(A)|$

$|\{0,1\}^{\{1,\ldots,n\}}| = |P(\{1,\ldots,n\})| = 2^n.$