Suma: $\kappa = |A|$, $\lambda = |B|$, $A \cap B = \emptyset \rightarrow \kappa + \lambda = |A \cup B|$

Iloczyn: $\kappa = |A|$, $\lambda = |B| \rightarrow \kappa \cdot \lambda = |A \times B|$

POTĘGOWANIE

<u>Def</u>. Jeśli $\kappa = |A|$ i $\lambda = |B|$, to

$$\kappa^{\lambda} = |A^B|.$$

Poprawność

Lemat: $\left.\begin{array}{l} A \sim C \\ B \sim D \end{array}\right] \longrightarrow A^B \sim C^D$

D-d.

$A^B \sim C^D$

$f: A \xrightarrow[\text{na}]{1-1} C$

$g: B \xrightarrow[\text{na}]{1-1} D$

Dla $\alpha \in A^B$ określamy

$$\Phi(\alpha) = f \circ \alpha \circ g^{-1}.$$

- $\Phi : A^B \to C^D$
- Φ jest 1-1 : zał.: że $\alpha, \beta \in A^B$ oraz $\Phi(\alpha) = \Phi(\beta)$ i wtedy $f \circ \alpha \circ g^{-1} = f \circ \beta \circ g^{-1}$

$$f\circ\alpha\circ g^{-1} = f\circ\beta\circ g^{-1} \quad /\, g$$

$$f\circ\alpha\circ(g^{-1}\circ g) = f\circ\beta\circ(g^{-1}\circ g)$$

$$f\circ(\alpha\circ \mathrm{id}_B) = f\circ(\beta\circ \mathrm{id}_B)$$

$$f^{-1}\,/\quad f\circ\alpha = f\circ\beta$$

$$(f^{-1}\circ f)\circ\alpha = (f^{-1}\circ f)\circ\beta$$

$$\mathrm{id}_A\circ\alpha = \mathrm{id}_A\circ\beta$$

$$\alpha = \beta \quad \blacksquare$$

- Φ jest "na" C^D

niech $\xi \in C^D$.

Określamy:

$$\alpha = f^{-1}\circ\xi\circ g$$

$$\begin{tikzcd} A \arrow[r, "f"] & C \\ B \arrow[r, "g"] & D \arrow[u, "\xi"'] \end{tikzcd}$$

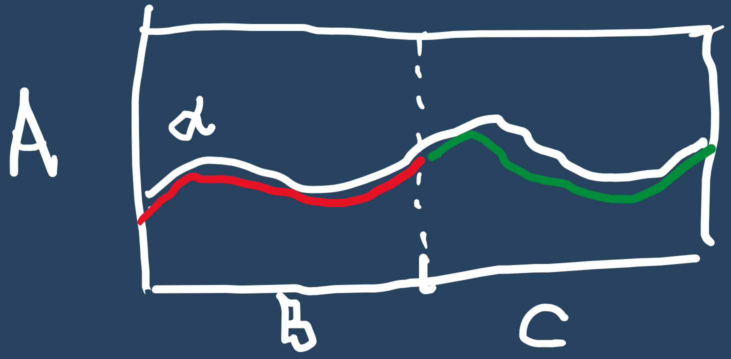
- $\alpha \in A^B$
- $\Phi(\alpha) = f\circ\alpha\circ g^{-1} =$

$$= f\circ(f^{-1}\circ\xi\circ g)\circ g^{-1}$$

$$= \mathrm{id}_C\circ\xi\circ\mathrm{id}_D = \xi \quad \blacksquare$$

Tw. $\boxed{\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}}$

D-d. Niech $|A| = \kappa$, $|B| = \lambda$, $|C| = \mu$, $B \cap C = \emptyset$.



Musimy pok. że

$$A^{B\cup C} \sim A^{B} \times A^{C}$$

Niech $\Phi(\alpha) = (\alpha \restriction B, \alpha \restriction C)$ dla $\alpha \in A^{B \cup C}$.

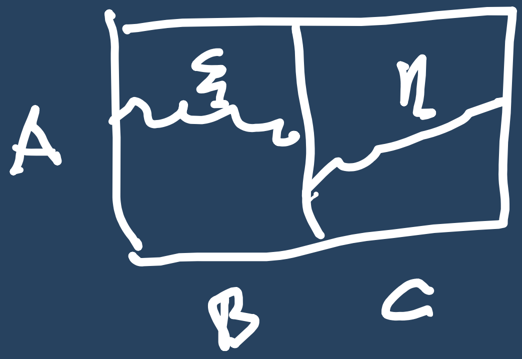
- $\Phi : A^{B\cup C} \longrightarrow A^{B} \times A^{C}$
- Φ jest 1-1 : $\alpha \neq \beta$; jest $x \in B \cup C$ t.że $\alpha(x) \neq \beta(x)$
  - $x \in B \longrightarrow \alpha \restriction B \neq \beta \restriction B$
  - $x \in C \longrightarrow \alpha \restriction C \neq \beta \restriction C$.

• Φ jest "na".

Niech $(\xi, \eta) \in A^B \times A^C$.

Niech $\alpha = \xi \cup \eta$.

Wtedy $\alpha \in A^{B \cup C}$ i $\Phi(\alpha) = (\xi, \eta)$.



$\blacksquare$

P

$$\mathfrak{c} \cdot \mathfrak{c} = 2^{\aleph_0} \cdot 2^{\aleph_0} \overset{!}{=}$$

$$= 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$$

$$\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$$

$\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$; $\mathfrak{c} = |\mathbb{R}|$

$\mathfrak{c} \cdot \mathfrak{c} = |\mathbb{R} \times \mathbb{R}|$

$\shortparallel$

$\mathfrak{c} = |\mathbb{R}|$

WNIOSEK $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$

UWAGA: $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

$\aleph_0 \cdot \aleph_0 = \aleph_0$

$\mathbb{C} = \{a + b \cdot i : a, b \in \mathbb{R}\}$

$a + bi \longleftrightarrow (a, b) \in \mathbb{R}^2$

$\mathbb{C} \sim \mathbb{R}$

$\mathbb{R}\times\mathbb{R} \sim \mathbb{R}$

$\mathbb{R}\times(\mathbb{R}\times\mathbb{R}) \sim \mathbb{R}\times\mathbb{R} \sim \mathbb{R}$

WNIOSEK: $(\forall n\in\mathbb{N})(|\mathbb{R}^n| = \mathfrak{c})$.

P ciągi

$\mathbb{R}^{\mathbb{N}}$

PÓŹNIEJ

Tw. $\boxed{(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}}$

Nast. godz: 12^{15}.

D-d. Ustalmy zbiory A, B, C t. że $|A| = \kappa$, $|B| = \lambda$, $|C| = \mu$.

Pok. że $(A^B)^C \sim A^{B \times C}$

Niech $\Phi : (A^B)^C \to A^{B \times C}$ będzie określona wzorem

$$\Phi(f)((b,c)) = f(c)(b). \qquad (f(c) : B \to A)$$

- Zał. że $f, g \in (A^B)^C$ i $\Phi(f) = \Phi(g)$, wtedy

$$(\forall b \in B)(\forall c \in C)\big(\Phi(f)((b,c)) = \Phi(g)((b,c))\big)$$

$$(\forall b \in B)(\forall c \in C)\big(\underbrace{f(c)}_{\in A^B}(b) = \underbrace{g(c)}_{\in A^B}(b)\big)$$

$(\forall c \in C)\,(f(c) = g(c))$ $\qquad f, g \in (A^B)^C$

zatem $f = g$.

ZATEM Φ jest 1-1.

$\Phi(f)((b,c)) = f(c)(b)$

• Φ jest "na".

Zał. że $\psi \in A^{B\times C}$.

niech $f(c) = \{(b, \psi((b,c))) : b \in B\}$.

dla $c \in C$.

wtedy $\Phi(f)((b,c)) = (f(c))(b) = \psi((b,c))$.

Więc $\Phi(f) = \psi$ $\quad\boxtimes$

$$\boxed{(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \cdot \mu}}$$

P CIĄGI LICZB RZECZYWISTYCH:

$$|\mathbb{R}^{\mathbb{N}}| = \mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} \overset{!}{=} 2^{\aleph_0 \cdot \aleph_0} =$$

$$= 2^{\aleph_0} = \mathfrak{c}$$

P $$|\mathbb{R}^{\mathbb{R}}| = \mathfrak{c}^{\mathfrak{c}} = (2^{\aleph_0})^{\mathfrak{c}} = 2^{\aleph_0 \cdot \mathfrak{c}} = 2^{\mathfrak{c}}$$

$$= |P(\mathbb{R})| > |\mathbb{R}| = \mathfrak{c}$$

$$\mathfrak{c} = 1 \cdot \mathfrak{c} \leq \aleph_0 \cdot \mathfrak{c} \leq \mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$$

$\mathfrak{c} \leq \aleph_0 \cdot \mathfrak{c} \leq \mathfrak{c}$ TW. C-B : $\aleph_0 \mathfrak{c} = \mathfrak{c}$

$$|\mathbb{R}^{\mathbb{R}}| = 2^{\mathfrak{c}} > \mathfrak{c}.$$

$$|\mathbb{R}^{\mathbb{N}}| = \mathfrak{c}$$

P $C(\mathbb{R},\mathbb{R}) = \{ f \in \mathbb{R}^{\mathbb{R}} : f \text{ jest ciągła} \}.$

$|C(\mathbb{R},\mathbb{R})| = ?$

- $\{\mathrm{const}_a : a \in \mathbb{R}\} \subseteq C(\mathbb{R},\mathbb{R})$

  $$\mathfrak{c} \leq |C(\mathbb{R},\mathbb{R})|.$$

- Pok. że $|C(\mathbb{R},\mathbb{R})| \leq \mathfrak{c}.$

Niech $\Phi : C(\mathbb{R},\mathbb{R}) \longrightarrow \mathbb{R}^{\mathbb{Q}}$ będzie określ.

$$\Phi(f) = f\restriction\mathbb{Q}.$$

FAKT: Φ jest 1-1.

Zał. że $f,g \in C(\mathbb{R},\mathbb{R})$ i $\Phi(f)=\Phi(g)$.

czyli $f\restriction\mathbb{Q} = g\restriction\mathbb{Q}$.

Niech $x \in \mathbb{R}$.

niech $(q_n)_{n\in\mathbb{N}}$ będzie ciągiem l. wym. zbieżnym do x, np.

$$q_n = \frac{\lfloor n\cdot x\rfloor}{n} \qquad \text{// zadanie!!!}$$

$\lim_n q_n = x$, $q_n \in \mathbb{Q}$

Wtedy

$$f(x) = f\left(\lim_n q_n\right) \overset{\text{ciągł}}{=} \lim_n f(q_n) = \lim_n (f\restriction\mathbb{Q})(q_n) = \lim_n (g\restriction\mathbb{Q})(q_n) = \lim_n g(q_n) = g\left(\lim_n q_n\right) = g(x).$$

$\blacksquare$

CZYLI: $\underline{f = g}$

$$\Phi : C(\mathbb{R},\mathbb{R}) \xrightarrow{1-1} \mathbb{R}^{\mathbb{Q}}$$

zatem

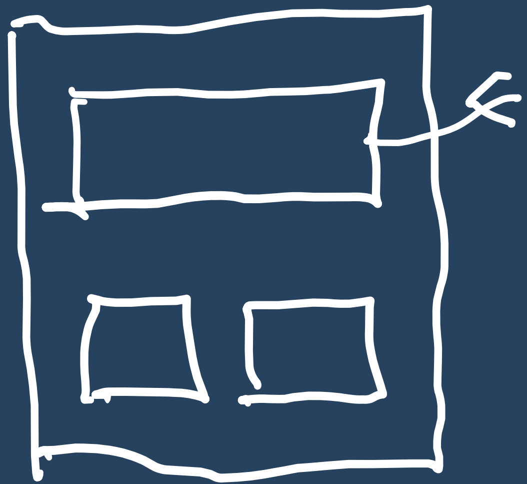
$$|C(\mathbb{R},\mathbb{R})| \leq |\mathbb{R}^{\mathbb{Q}}| = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$$

Zatem:

$$|C(\mathbb{R},\mathbb{R})| = \mathfrak{c}.$$

(P) Currying

strony www:



<div> ... </div>

<div id = "m1"> ... </div>

CSS ← służą do dekoracji

Zadanie: narzędzia w JavaScript do kolorowania elementów

```
function CEL(elem, cb, ct){
    elem.style.backgroundColor = cb;
    elem.style.color = ct;
}
```

$$X^{E \times C \times C} \sim \left(X^{E}\right)^{C \times C}$$

CURRYING :

$$F : A \times B \to C \qquad F : A \times B \times C \to D$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\tilde{F} : A \to (B \to C) \qquad \tilde{F} : A \to (B \to (C \to D))$$

Haskell : wszystkie funkcje są jednej zmiennej.

add x y : x + y

add : Int → (Int → Int)

Haskell Curry ← mat. amerykański XX wieku

"progr. funkc."

Teoria Kategorii