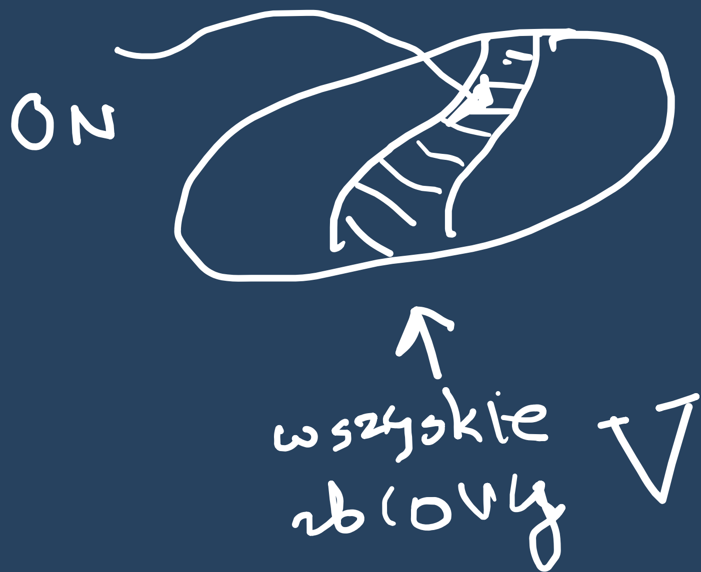
Tw. $(\forall \alpha,\beta)\big((\mathrm{ord}(\alpha) \wedge \mathrm{ord}(\beta)) \to (\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha)\big)$

Oznaczenia: 1) $\mathrm{ord}(x) \equiv$ "$x \in ON$"

2) $(x = x) \equiv$ "$x \in V$"




FAKT: ON nie jest zbiorem

D-d. Zał. że ON jest zbiorem.

Niech $\gamma = \bigcup ON$. Wtedy $\bullet\ \gamma \in ON$.

$\bullet\ (\forall \alpha)(\alpha \in ON \to \alpha \leq \gamma)$


Wtedy $\gamma + 1\ (= \gamma \cup \{\gamma\}) \in ON \quad \leq \gamma$

więc $\gamma \in \gamma$ ■

Tw (o sztywności liczb porz).

Zał. że $\alpha, \beta \in ON$ oraz, że $f : (\alpha, \leq_\alpha) \xrightarrow{IZO} (\beta, \leq_\beta)$

Wtedy $\alpha = \beta$ i $f = id_\alpha$.

(P) $f : x \to x+1$, $x \in \mathbb{Q}$; $f : (\mathbb{Q}, \leq) \xrightarrow{IZO} (\mathbb{Q}, \leq)$

D-d. Zał. że $f : \alpha \xrightarrow{IZO} \beta$.

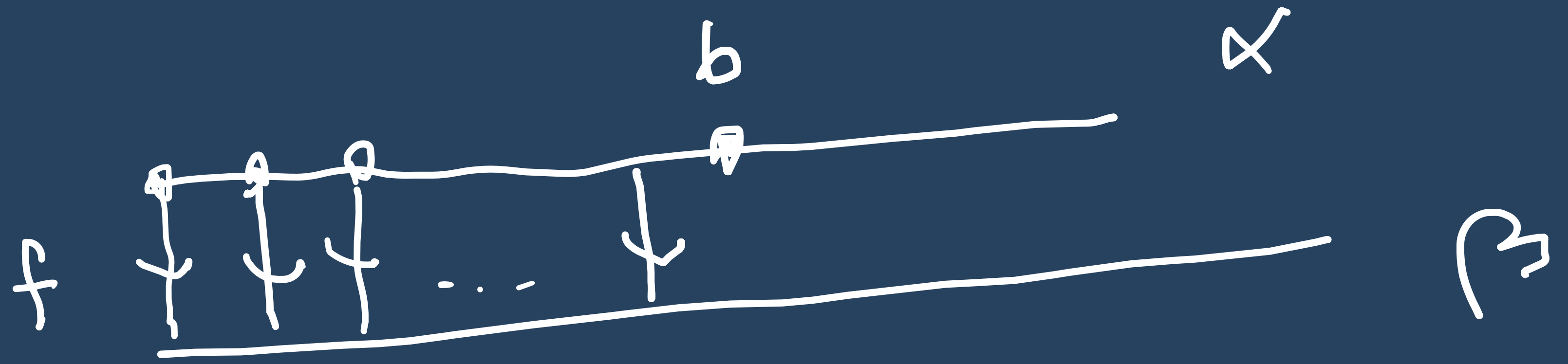
Niech $D = \{x \in \alpha : f(x) = x\}$

CLAIM: $D = \alpha$.

Zał. że $D \neq \alpha$. Niech $B = \{x \in \alpha : f(x) \neq x\}$.

Wtedy $B \neq \emptyset$. Niech $b = \leq_\alpha\text{-}\min(B)$.

$f(0) = 0$
$f(1) = 1$
$\vdots$



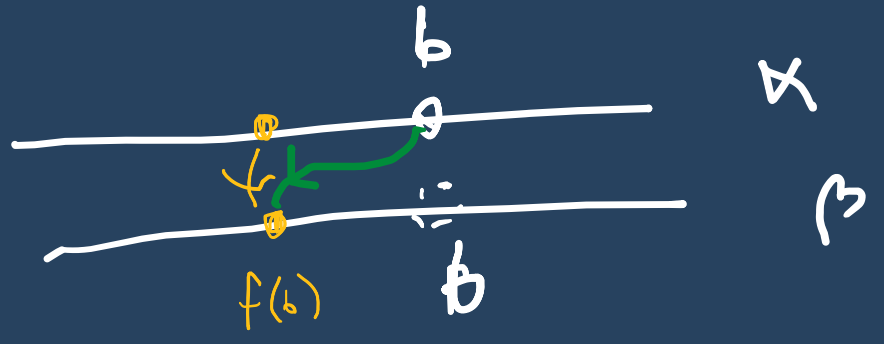

- $x<b \longrightarrow f(x)=x$
- $f(b) \neq b$.

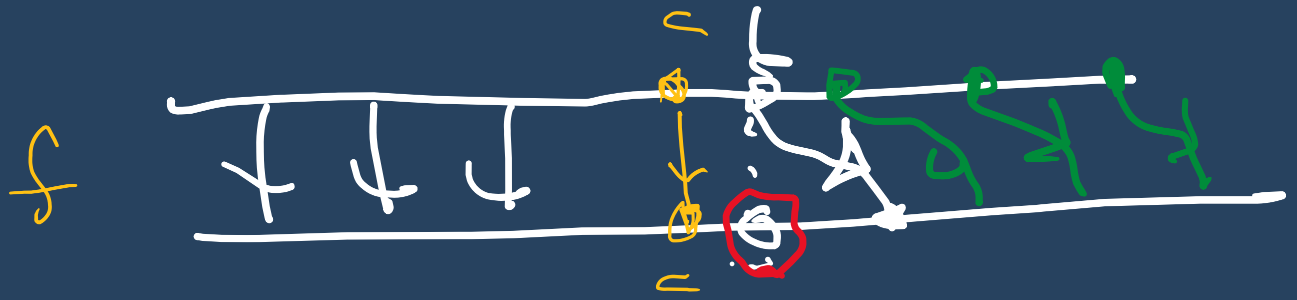
gdyby $f(b) < b$

$f(f(b)) = f(b)$

to $f$ nie byłaby 1–1.

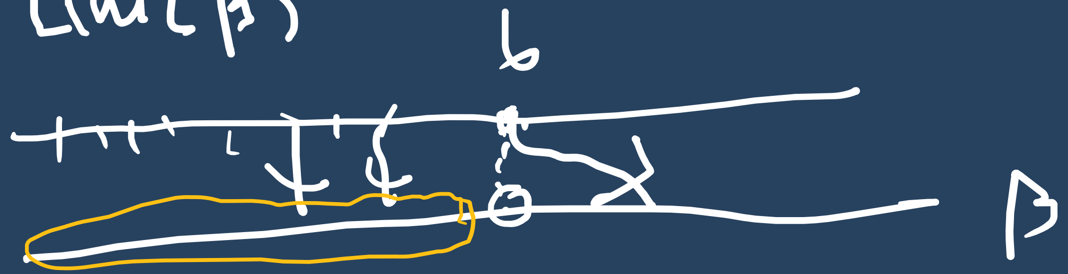
ZATEM $b < f(b)$.





- C1 : $b = c+1$ : wtedy $f(b) > c+1 = b$
  więc $b \notin \mathrm{rng}(f)$. sprz

- C2 : $\mathrm{Lim}(\beta)$

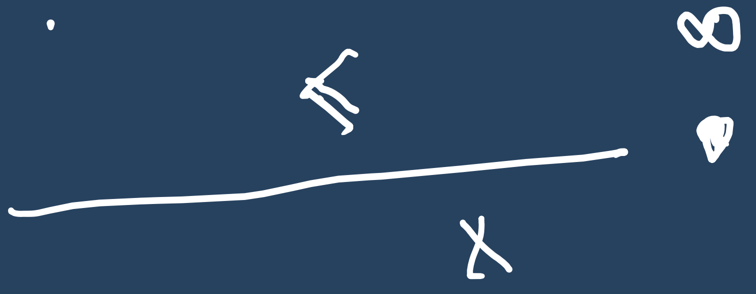


$b = \bigcup\{f(x) : x < b\} = \bigcup\{x : x < b\}$

znowu $b \notin \mathrm{rng}(f)$ ☒

**Tw.** Niech $(X, \leq)$ będzie dobrym porządkiem. Istnieje dokładnie jedna $\alpha \in ON$ t. że

$$(X, \leq) \overset{\text{izo}}{\cong} (\alpha, \leq_\alpha).$$



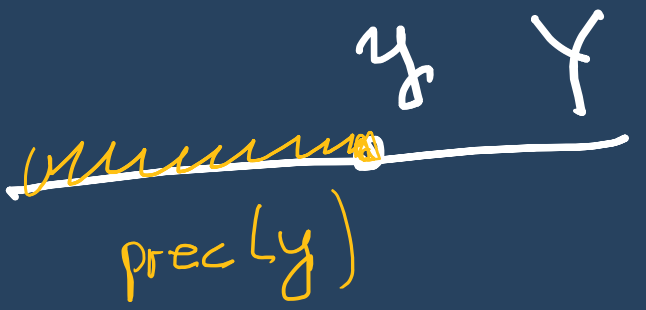

**D-d.** Ustalmy d. porz. $(X, \leq)$.

Niech $\infty \notin X$, $Y = X \cup \{\infty\}$

$$\leq_Y = \leq \cup (X \times \{\infty\}) \cup \{(\infty, \infty)\}.$$

Wtedy $(Y, \leq_Y)$ też jest d. porz.

$$\mathrm{prec}(y) = \{t \in Y : t \leq y\}$$



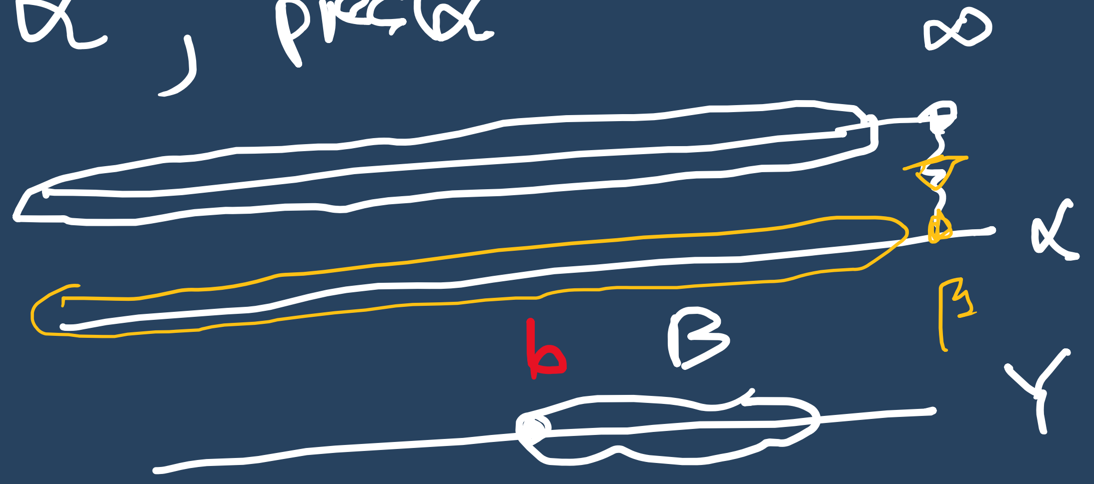
$D = \{ y \in Y : (\exists f)(\exists \alpha)( f : \mathrm{prec}(y) \xrightarrow{1\text{-}1,\ \text{na}} \alpha ) \}$

- Jeśli $D = Y$ to koniec, bo

  mamy $f : \mathrm{prec}(\infty) \to \alpha$, $\mathrm{prec}(\alpha)$

  $f(\infty) = \beta$

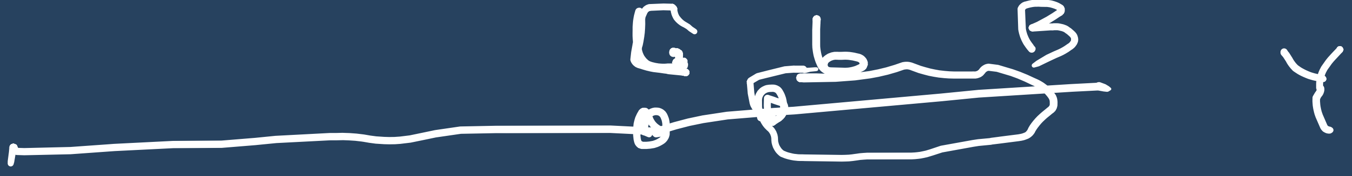
  $X \overset{1\text{-}1,\ \text{na}}{\simeq} \beta$




- $B = Y \setminus D$.

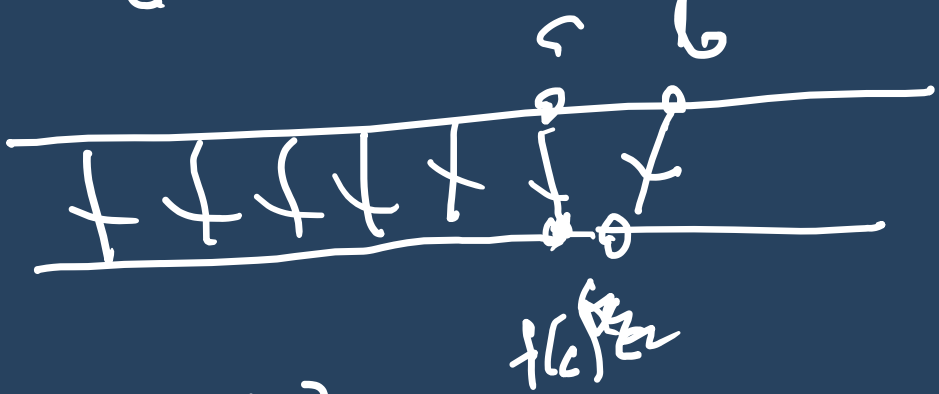
  $b = \leq_Y\text{-}\min(B)$

C1 ;




Jest $c \in Y$ t.że $c < b \wedge \neg(\exists z)(c < z < b)$.

wtedy $c \notin B$. Jest $f_c : \mathrm{prec}(c) \xrightarrow{\text{izo}} \alpha_c$



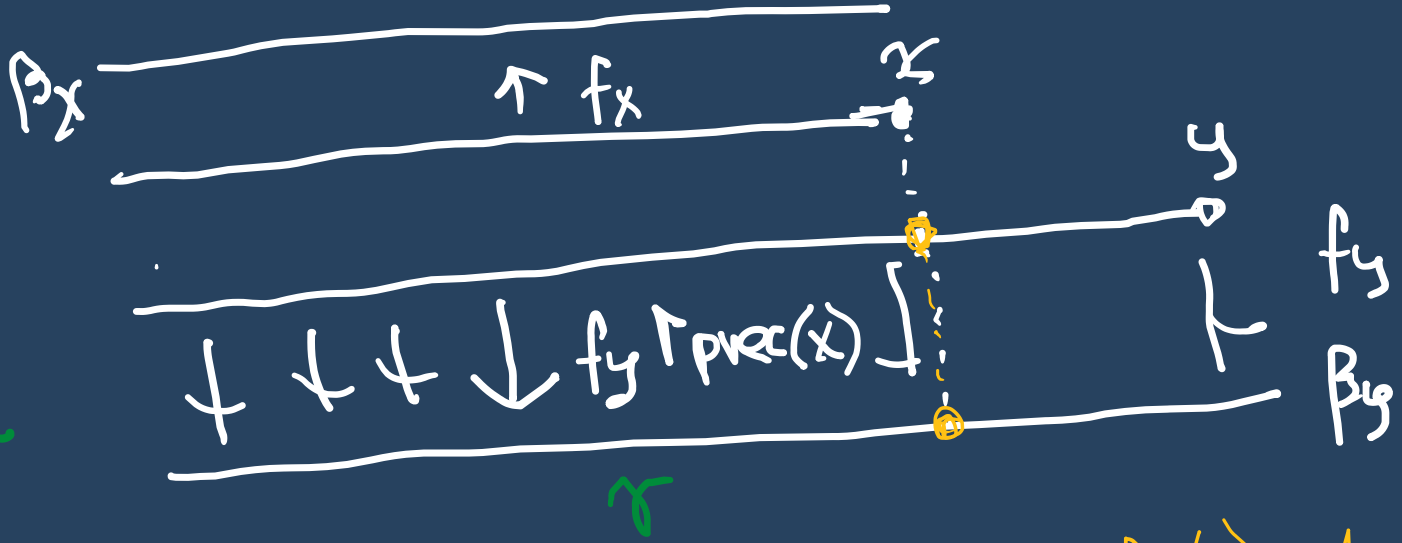

niech

$$f_b = f_c \cup \{(b, f_c(c)+1)\}.$$

C2. $\neg(\exists z<b)(\forall t)(z<t<b)$.


• Dla $y<b$ mamy $f_y : \mathrm{pred}(y) \xrightarrow{\mathrm{IZO}} \beta_y$

• $x<y<b$




$$\mathrm{IZO}\begin{cases} f_x : \mathrm{pred}(x) \longrightarrow \beta_x \\ \underbrace{f_y \upharpoonright \mathrm{pred}(x)}_{f_y^*} : \mathrm{pred}(x) \to \uparrow \end{cases}$$

$$f_y \upharpoonright \mathrm{pred}(x) : \mathrm{pred}(x) \longrightarrow \underbrace{f_y(x)+1}_{\gamma}$$

$$f_x \circ (f_y^*)^{-1} = \mathrm{id}_\gamma \; ; \; f_y^* \circ f_x^{-1} = \mathrm{id}_{\beta_x}$$

Cel: $x < y < b \longrightarrow f_y \restriction \mathrm{pred}(x) = f_x$

Kładziemy

$$f_b = \bigcup_{x<b} f_x \cup \{(b, \overbrace{\bigcup_{x<b} \mathrm{rng}(f_x)}^{\gamma})\}$$


$\{\mathrm{rng}(f_x) : x < b\}$

ON

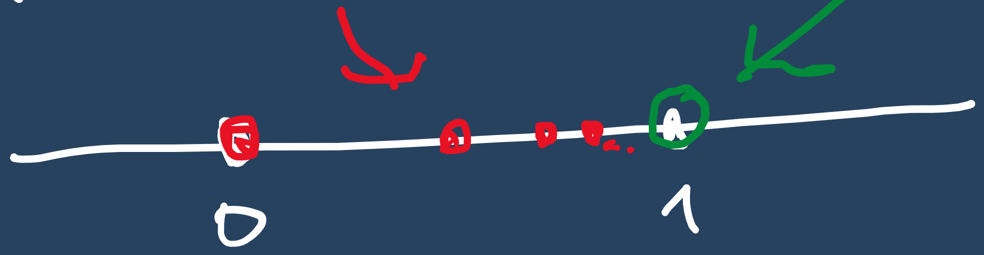
dow 2. Zatem $B = \emptyset$. $\square$

**Def.** Typem porządkowym cz. porz $(X, \le)$ nazywamy taką $\alpha \in ON$, że

$$(X, \le) \overset{izo}{\simeq} (\alpha, \le_\alpha).$$

Ozn. $\overline{(X, \le)}$ = typ porz. $(X, \le)$.

P: $\overline{\left(\left\{1 - \frac{1}{n+1} : n \in \mathbb{N}\right\} \cup \{1\}, \le\right)} = \omega \cup \{\omega\} = \omega + 1$



Tw (o indukcji pozaskończonej)

Zał. że $\varphi(x,y)$ jest formułą (z parametrami) t. że $(\forall x)(\exists! y)\,\varphi(x,y)$. Niech $\alpha \in ON$.

Istnieje dokładnie jedna funkcja $f : \alpha \longrightarrow V$ t. że

$$(\forall \beta < \alpha)\left(\varphi(f\restriction_\beta, f(\beta))\right).$$

P weźmy, że $\beta = 0$. $\quad f\restriction_\beta = \emptyset \quad f\restriction_1$

$f(0) : \quad \varphi(\emptyset, f(\emptyset)).$

weźmy $\beta = 1 = \{0\}$ ; $\quad \varphi(\{(0,f(0))\}, f(1))$

Rekursje $\begin{cases} f_0 = 0 \\ f_{n+1} = f_0 + \ldots + f_n + 1 \end{cases}$ $n \in \mathbb{N}$

Q: znajdź wzór na $f_n$.

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D-d. 1 najp. pok. jednozn.

mamy $f, g : \alpha \to V$ : $\forall \beta < \alpha$

$\varphi(f \upharpoonright \beta, f(\beta))$, $\varphi(g \upharpoonright \alpha, g(\alpha))$

Pok. że $f = g$.

Wpk. niech $\beta = \min\{\xi < \alpha : f(\xi) \neq g(\xi)\}$

$\forall x \exists! y\ \varphi(x, y)$.

2) Rozw. $\beta = \min\{\gamma < \alpha :$ nie ma takiej funkcji $\}$

C1 $\mathrm{SCC}(\beta)$

C2 $\lim(\beta)$ zadanie

P Def. rek. por.

$$\lim(\lambda): \begin{cases} \alpha + 0 = \alpha \\ \alpha + (\beta+1) = (\alpha+\beta)+1 \\ \alpha + \lambda = \bigcup_{\xi<\lambda} (\alpha+\xi) \end{cases}$$

i dziedzinie

Sprow. do indukcji.

$\varphi(x, y)$ : 1) $\neg$(x jest funkcją o wart. w ON) $\to y = \emptyset$

2) $x$ - funkcja $\mathrm{dom}(x) = \beta+1$, $y \to f(\beta)+1$

3) $x$ - funkcja, $\mathrm{dom}(x) \in \mathrm{LIM}$ $\longrightarrow y = \bigcup \mathrm{Rng}(x)$.

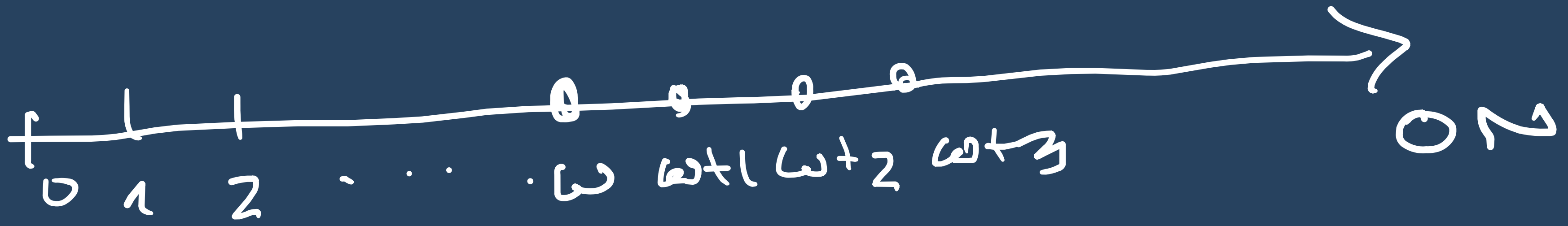
P

$$3+\omega = \bigcup_{n\in\omega}(3+n) = \omega$$

$$\omega + 3 = (\omega+2)+1 = ((\omega+1)+1)+1$$

$$= (((\omega+0)+1)+1)+1 =$$

$$= ((\omega+1)+1)+1 .$$



**Def.**

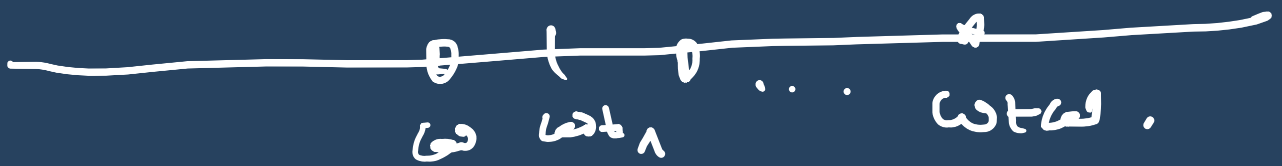
$$\begin{cases} \alpha \cdot 0 = 0 \\ \alpha \cdot (\beta+1) = (\alpha \cdot \beta) + \alpha \\ \alpha \cdot \lambda = \bigcup\limits_{\beta<\lambda} (\alpha \cdot \beta) \end{cases}$$

P

$$2 \cdot \omega = \bigcup_{n<\omega} (2 \cdot n) = \omega$$

$$\omega \cdot 2 = \omega(1+1) = (\omega \cdot 1) + \omega$$

$$= \omega + \omega$$

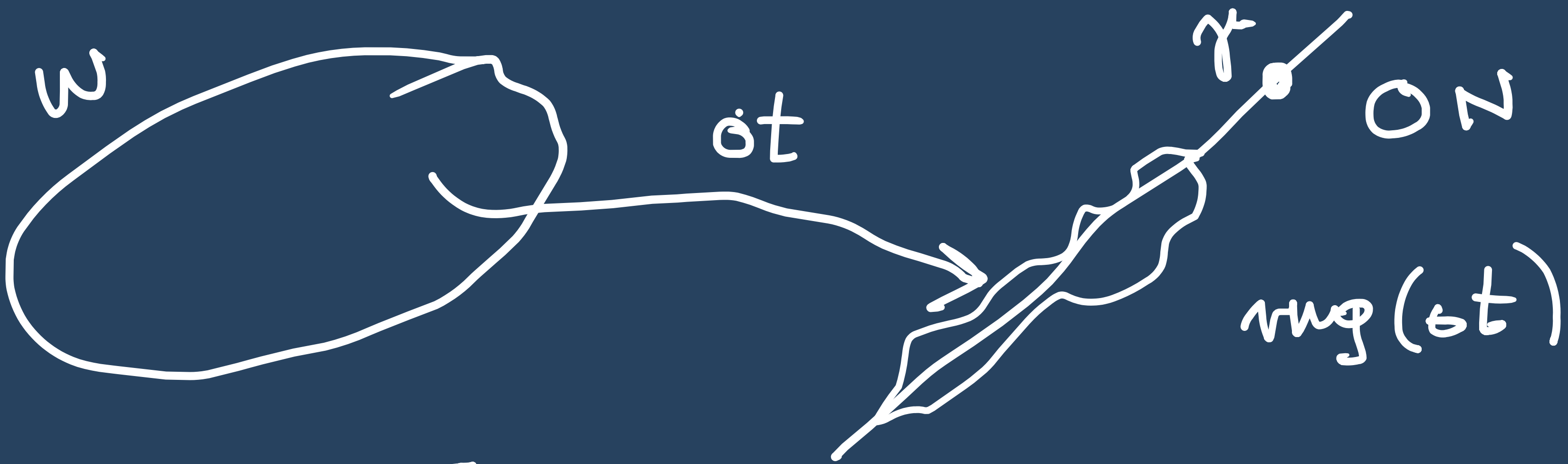


Weźmy dowolny zbiór $X$.

$$W = \{ r \subseteq X \times X : r \text{ dobre porz. swojej dziedziny} \}$$

Dla $r \in W$ : $\alpha_r$ = typ porządkowy $r$.

$$ot : W \longrightarrow ON : r \longmapsto \alpha_r .$$




AKSJ. ZASTĘPOWANIA.

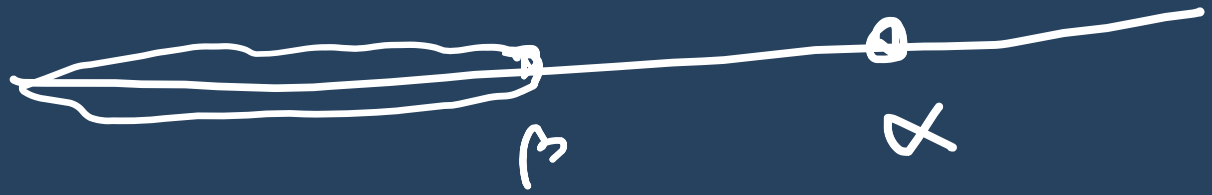
Jest $\gamma \in ON$ t.że $(\forall r \in W)(ot(r) < \gamma)$.

Def. $\mathcal{H}(x) = \min\{\gamma \in ON : \neg(\exists f)(f : \gamma \xrightarrow{1-1} X)\}$

(funkcja Hartogs'a).

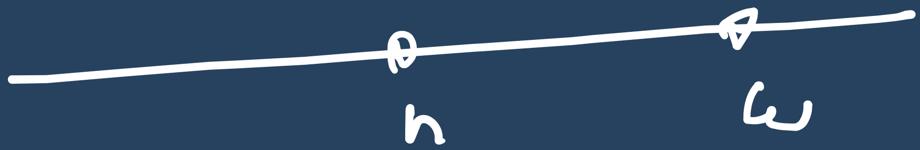

nie ma takiej funkcji

**Def.** $\mathrm{card}(\alpha) \equiv \mathrm{ord}(\alpha) \wedge (\forall \beta < \alpha)(|\beta| < |\alpha|)$

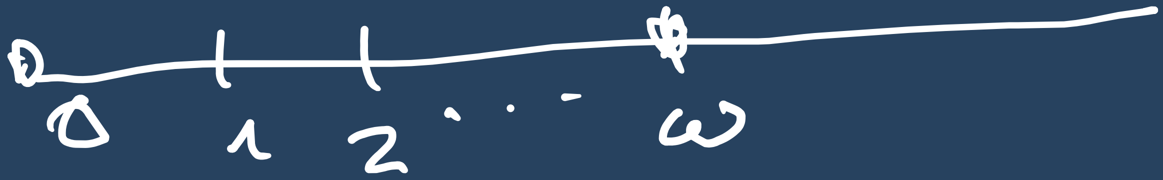



Propozit.

- $\mathrm{card}(\omega)$




- $(\forall n \in \omega)\ \mathrm{card}(n)$



$$\begin{cases} \aleph_0 = \omega \\ \aleph_{\alpha+1} = \mathcal{H}(\aleph_\alpha) \\ \aleph_\lambda = \bigcup\limits_{\xi<\lambda} \aleph_\xi \quad \leftarrow \quad \mathrm{Lim}(\lambda) \end{cases}$$

$|\aleph_1| > \omega$

Co to jest $\aleph_1$ ?


$$\omega \leq \beta < \aleph_1 \longrightarrow |\beta| = \aleph_0 .$$

$\aleph_1$ = najmn. l. kard. $> \aleph_0$

= najmn. l. kard. nieprzeliczalna

$$CH = \boxed{2^{\aleph_0} = \aleph_1}$$

Continuum Hypothesis.

$$CH \equiv (\forall A \subseteq \mathbb{R})(|A| < \mathfrak{c} \to |A| \leq \aleph_0).$$

Tw (Gödel) $\mathrm{Cons}(ZF) \to \mathrm{Cons}(ZFC + CH)$

Tw (Cohen) $\mathrm{Cons}(ZF) \to \mathrm{Cons}(ZFC + \neg CH)$

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KONIEC WYKŁADU.