ZASADA WŁĄCZ. – WYŁĄCZANIA

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{T \subseteq \{1,\dots,n\} \\ |T| = k}} |A_T|,$$

gdzie $$A_T = \bigcap_{i \in T} A_i .$$

PRZYKŁAD: $n = 3$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - \\ - \left(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\right) + |A_1 \cap A_2 \cap A_3|$$

Ile elem. ma zbiór:

$$\{n \in \{1, .., 100\} : 2 | n \vee 3 | n \vee 5 | n\} \,?$$

$$A_1 = \{n \in [100] : 2 | n\}$$

$$A_2 = \{n \in [100] : 3 | n\}$$

$$A_3 = \{n \in [100] : 5 | n\}$$

FAKT: $\left| \{k \in \{1, .., n\} : a | k\} \right| = \left\lfloor \frac{n}{a} \right\rfloor$ Z

FAKT:

- $x - 1 < \lfloor x \rfloor \leq x$
- $\lfloor x \rfloor \in \mathbb{Z}$

$$|A_1| = \left\lfloor \frac{100}{2} \right\rfloor = 50$$

$$|A_2| = \left\lfloor \frac{100}{3} \right\rfloor = 33$$

$$|A_3| = 20$$

$$|A_1 \cap A_2| = \left\lfloor \frac{100}{6} \right\rfloor = 16$$

$$\vdots$$

$$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{100}{30} \right\rfloor = 3$$

$$|\ldots| = 50+33+20-(\quad)+3 = 74$$

$$\left\{ \begin{array}{c} a \mid n \wedge b \mid n \\ \Updownarrow \\ NWW(a,b) \mid n \end{array} \right.$$

Sprawdzenie (Python)

"list comprehension"

```
>>> X = [x for x in range(1,101) if(x%2 ==0)
            or (x%3 ==0) or (x%5 ==0)]
>>> len(X)
>>> 74
```

D-d (Zw-w).

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{\substack{T \subseteq \{1..n\} \\ T \neq \emptyset}} (-1)^{|T|+1} |A_T|$$

$T \in P_+([n]) = \{X \subseteq \{1..n\} : X \neq \emptyset\}$.

- dla $(n=2)$ wzór jest prawdz.
- założ. że dla n jest ok.

$$\left|\bigcup_{k=1}^{n+1} A_k\right| = \left|\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right| =$$

$$\left|\bigcup_{k=1}^{n} A_k\right| + |A_{n+1}| - \left|\left(\bigcup_{k=1}^{n} A_k\right) \cap A_{n+1}\right| =$$

$$= \underline{\qquad}\ |\ |\ \underline{\qquad\qquad} - \left|\bigcup_{k=1}^{n} \underbrace{(A_k \cap A_{n+1})}_{A_k^*}\right|$$

$$= \sum_{T \in P_t([n])} (-1)^{|T|+1} |A_T| + |A_{n+1}| - \sum_{T \in P_t([n])} (-1)^{|T|+1} |A_T^*| =: (*)$$

$$A_T^* = \bigcap_{i \in T} A_i^* = \bigcap_{i \in T} (A_i \cap A_{n+1}) =$$

$$= \left(\bigcap_{i \in T} A_i \right) \cap A_{n+1} =$$

$$= \bigcap_{i \in T \cup \{n+1\}} A_i = A_{T \cup \{n+1\}}$$

$$(*) = \sum_{T \in P_t([n])} (-1)^{|T|+1} |A_T| + \boxed{|A_{n+1}|} - \sum_{T \in P_t([n])} (-1)^{|T|+1} |A_{T \cup \{n+1\}}| \qquad (**)$$

$$(**) = \sum_{\substack{T \in P([n+1]) \\ n+1 \in T \\ T \cap [n] \neq \emptyset}} (-1)^{|T|+1} |A_T|$$

$$= \sum_{T \in P_+([n])} (-1)^{|T|+1} |A_T| + \sum_{\substack{T \in P_+([n+1]) \\ n+1 \in T}} (-1)^{|T|+1} |A_T|$$

ALE: $P([n+1]) = P([n]) \cup \{T \in P([n+1]) : n+1 \in T\}$

$$= \sum_{T \in P_+([n+1])} (-1)^{|T|+1} |A_T| \quad \blacksquare$$

NIEPORZĄDKI

ustalony $n \in \mathbb{N}^+$.

Sym_n = zbiór wszystkich perm. $\{1, \dots, n\}$

- $|Sym_n| = n!$

$D_n = \{\pi \in Sym_n : (\forall i \in \{1..n\})(\pi(i) \neq i)\}$.

$Fix_i^{(n)} = \{\pi \in Sym_n : \pi(i) = i\}$

$D_n = Sym_n \setminus \bigcup_{i=1}^{n} Fix_i^{(n)}$

$$\left|\bigcup_{i=1}^{n} \mathrm{Fix}_i^{(n)}\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{T\in P([n]) \\ |T|=k}} \left|\mathrm{Fix}_T^{(n)}\right| = (*)$$

$$\left|\mathrm{Fix}_T^{(n)}\right| = \left|\bigcap_{i\in T} \mathrm{Fix}_i^{(n)}\right| = \left|\{\pi\in \mathrm{Sym}_n : (\forall i\in T)(\pi(i)=i)\}\right|$$

$\{1,\dots,n\}$

T

$$= (n-|T|)!$$

$$(*) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{T \subseteq \{1..n\} \\ |T| = k}} (n-|T|)! =$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)! = \sum_{k=1}^{n} (-1)^{1+k} \frac{n!}{k!\,\cancel{(n-k)!}} \cancel{(n-k)!}$$

$$= n! \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!} .$$

$$|D_n| = n! - n! \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!}$$

$$= n! \left(1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!} \right)$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} .$$

$$\frac{|D_n|}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

ANALIZA: $e^x = \sum_{k \geq 0} \frac{x^k}{k!}$ $(x \in \mathbb{R})$

szybko zbieżne.

WNIOSEK: $e^{-1} = \sum_{k \geq 0} \frac{(-1)^k}{k!}$

WNIOSEK: $\lim_{n \to \infty} \frac{|D_n|}{n!} = \frac{1}{e} \approx \frac{1}{3}.$

DZIELNIKI LICZB.

$$n = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k} \in \mathbb{N}^+$$

$p_1 < p_2 < \ldots < p_k$ liczby pierwsze

$\alpha_i > 0$; $i = 1 \ldots k$.

- $d \mid n \iff d = p_1^{\beta_1} \cdot \ldots \cdot p_k^{\beta_k} \wedge (0 \le \beta_1 \le \alpha_1) \wedge \ldots \wedge (0 \le \beta_k \le \alpha_k)$

WNIOSEK: $\left|\{d \in \mathbb{N} : d \mid n\}\right| = \prod_{i=1}^{k} (1 + \alpha_i)$.

- $\varphi(n) = |\{k \in [n] : nwd(k,n) = 1\}|$

Funkcja "psi" Eulera.

- $k < n$: $NWD(k,n) > 1 \equiv (\exists i \in [k])(p_i \mid k)$.

- $D_i = \{k \le n : 1 \le k \wedge p_i \mid \overset{k}{\cancel{n}}\}$ TU BYŁ BŁAD

- $\left|\bigcup_{i=1}^{k} D_i\right|$ ← zasada wł. wył.

$$n = \prod_{i=1}^{k} p_i^{\alpha_i} \qquad \underline{\alpha_i > 0}.$$

$$|D_1 \cap D_3| = |\{k \in [n] : p_1 | k \wedge p_3 | k\}|$$

$$= |\{k \in [n] : p_1 \cdot p_3 | k\}| =$$

$$= \left\lfloor \frac{n}{p_1 \cdot p_3} \right\rfloor = \frac{n}{p_1 \cdot p_3}$$

Σ .....

$$\left(\frac{n}{p_1} + \frac{n}{p_2} + \dots + \frac{n}{p_k}\right) - \left(\frac{n}{p_1 \cdot p_2} + \frac{n}{p_1 \cdot p_3} + \dots\right)$$

$$+ \left(\frac{n}{p_1 p_2 p_3} + \dots\right) - \dots$$

$$\varphi(n) = n\left(1 - \left(\frac{1}{p_1} + \dots + \frac{1}{p_k}\right) + \left(\frac{1}{p_1 p_2} + \dots\right) - (\ )\right)$$

$$\overset{!}{=} n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

$$(1-x)(1-y)(1-z) = 1 - (x+y+z) + (xy + xz + yz)$$

$$- x \cdot y \cdot z.$$

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

$\boxed{Z}$ • $\text{nwd}(n, m) = 1 \longrightarrow \varphi(n \cdot m) = \varphi(n) \cdot \varphi(m)$.

• $\varphi(p) = p - 1$

• Z $\varphi(p^k) = ?$