Wzór dwumianowy:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}. \qquad (*)$$

Co jest potrzebne?

- $x+y = y+x$ przem. dod. dla dow. $x, y$
- łączność dod.
- łączn. mnożenia
  przemienność mnoż. $x$ i $y$

$(*)$ jest prawdziwa w pierścieniu $(R, +, \cdot)$

jeśli $x \cdot y = y \cdot x$.

Subtelność: $(R,+,\cdot)$ ← pierścień

$x \in R$, $n \in \mathbb{N}$

co znaczy $n \cdot x$ ?

$$n \cdot x = \underbrace{x + x + \ldots + x}_{n}.$$

(P) $\mathbb{Z}_5 = (\{0, \ldots, 4\}, +_5, \cdot_5)$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

$$(x+y)^5 = x^5 + \underbrace{5 \cdot x^4 y + 10 \cdot x^3 y^2 + 10 \cdot x^2 y^3 + 5 \cdot x^1 \cdot y^4}_{0} + y^5$$

$$(x+y)^5 = x^5 + y^5$$

$\textcircled{P}$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$$

$$(1+x)^n + (1-x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \left(1 + (-1)^k\right)$$

$$= \sum_{k} \binom{n}{2k} x^{2k} \cdot 2$$

$$\sum_{k} \binom{n}{2k} x^{2k} = \frac{1}{2}\left((1+x)^n + (1-x)^n\right)$$

$x \leftarrow 1$

$$\sum_k \binom{n}{2k} = \frac{1}{2}\left(2^n + 0^n\right) = \frac{1}{2}\left(2^n + [\![n=0]\!]\right)$$

$$\sum_k \binom{n}{2k} = 2^{n-1} + \frac{1}{2}[\![n=0]\!]$$

$$\left|\left\{T \in P([n]) : 2 \mid |T| \right\}\right|$$

$$\frac{\sum_k \binom{n}{2k}}{2^n} = \frac{1}{2} + \frac{1}{2^{n+1}}[\![n=0]\!]$$

Q: $\sum_k \binom{n}{3k} = ?$

Liczby $\{-1, 1\}$ : pierw. 2 stopnia z $1$.

$x^2 = 1$ $\quad || \ \mathbb{C}$


Pobawcie się pierw. $\quad x^3 = 1$ w $\mathbb{C}$

$(1+\xi)^n =$

$(1+\xi^2)^n =$

$(1+\underset{\uparrow\,\xi^0}{1})^n =$

# Wzory Vandermonda.

$$(1+x)^n(1+x)^m = (1+x)^{n+m} = \sum_k \binom{n+m}{k} x^k$$

$$\parallel$$

$$\left(\sum_{k=0}^{n} \binom{n}{k} x^k\right) \cdot \left(\sum_{k=0}^{m} \binom{m}{k} x^k\right)$$

$$(a_0 + a_1 x + \dots + a_n x^n)(b_0 + b_1 x + \dots + b_m x^m)$$

$$= (a_0 b_0) + \underline{(a_0 b_1 + a_1 b_0)} x +$$

$$(a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \dots$$

$$\dots + a_n b_m x^{n+m}$$

$$a_0 + a_1 x + \dots + a_n x^n =$$

$$= a_0 + \dots + a_n x^n + 0 x^{n+1} + 0 x^{n+2} + \dots +$$

$$+ 0 \cdot x^{n+m}$$

$$\left(\sum_{k=0}^{n} a_k x^k\right)\left(\sum_{k=0}^{m} b_k x^k\right) =$$

$$= \sum_{k=0}^{n+m} \left(\sum_{\substack{i+j=k \\ i,j \geq 0}} a_i \cdot b_j\right) \cdot x^k =$$

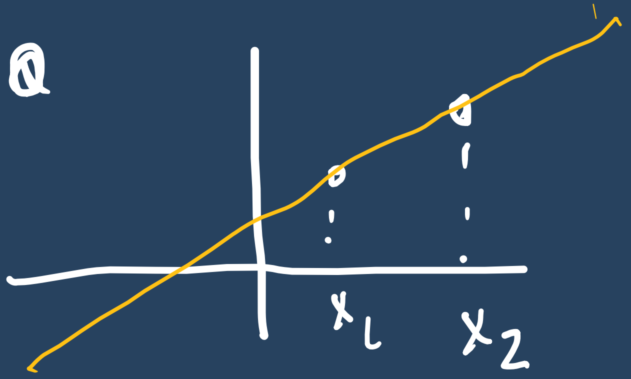
$$= \sum_{k=0}^{n+m} \left(\sum_{i=0}^{k} a_i b_{k-i}\right) x^k$$

$$\sum_{k=0}^{n+m} \binom{n+m}{k} x^k = \sum_{k=0}^{n+m} \left( \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} \right) x^k \quad (**)$$

wielomian stopnia $n+m$      wielomian stopnia $n+m$.

<u>Tw.</u> Zał., że $K$ jest ciałem nieskończ.
Zał., że $w, v \in K[x]$ oraz $\deg(w), \deg(v) \leq N$.
Zał., że $x_1, \ldots, x_{N+1} \in K$ są parami różne
oraz $w(x_1) = v(x_1), \ldots, w(x_{N+1}) = v(x_{N+1})$.
WTEDY $w = v$.

UWAGA : w $\mathbb{R}$

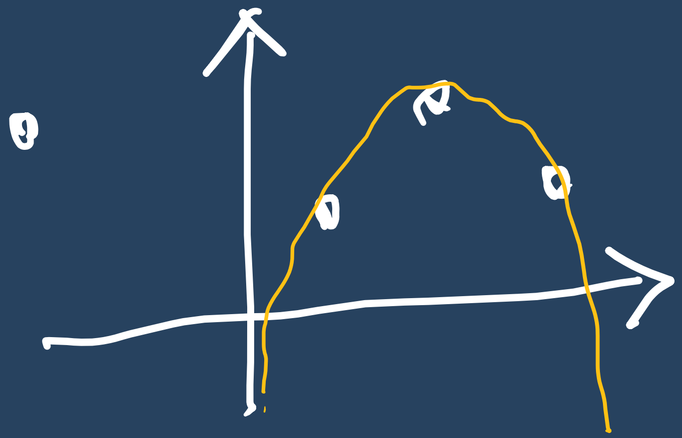



$w, v \in \mathbb{R}[x]$
stopnia $\leq 1$

$w(x) = a + bx$
$v(x) = c + dx$

$$\left.\begin{aligned} w(x_1) &= v(x_1) \\ w(x_2) &= v(x_2) \end{aligned}\right\} \to w = v$$

$x_1 \neq x_2$



parabole

!! $\left\{\begin{array}{l} \text{wzór interpolacyjny} \\ \text{Lagrange'a} \end{array}\right.$

<u>D-d.</u> Niech $\varphi(x) = W(x) - V(x)$.

- $\varphi$ jest stopnia $\leq N$
- $\varphi(x_1) = \ldots = \varphi(x_{N+1}) = 0$

więc: $\varphi(x) = C \cdot \underbrace{(x-x_1) \cdot \ldots \cdot (x-x_{N+1})}_{\text{wiel. stopnia } N+1}$

zatem $C = 0$.

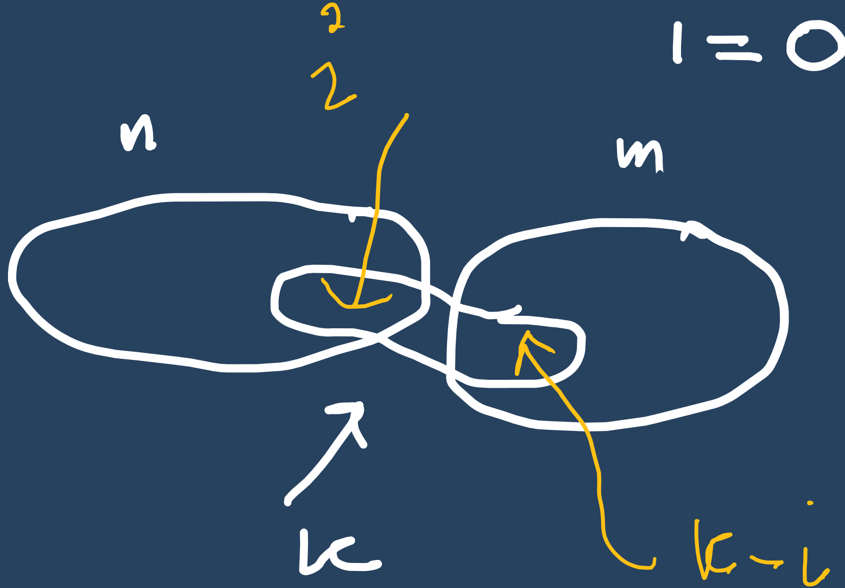
zatem $W = V$. $\square$

**Tw.** $\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \cdot \binom{m}{k-i}$

Alternatywny dowód:

$$\left| [n+m]^k \right| = \left| \bigcup_{i=0}^{k} \{ T \in [n+m]^k : |T \cap [n]| = i \} \right|$$

$$= \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}.$$



$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$$

$m, k \leftarrow n$

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{i}$$

$$\boxed{\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}}$$

Tożsamość Cauchy'ego.

# UOGÓLNIENIE

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{\prod_{i=0}^{k-1}(n-i)}{k!}$$

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!} = \frac{\prod_{i=0}^{k-1}(x-i)}{k!}$$ ← wielomian z $\mathbb{C}[x]$ stopnia $k$

$$\binom{-1}{k} = \frac{1}{k!}(-1)(-2)\cdot\ldots\cdot((-1)-(k-1))$$

$$= \frac{1}{k!}(-1)(-2)\cdot\ldots\cdot(-k) = \frac{1}{k!}(-1)^k k!$$

$$\binom{-1}{k} = (-1)^k .$$

$$\binom{\alpha}{k} = \frac{1}{k!}\,\overbrace{\alpha\cdot(\alpha-1)\cdot\ldots\cdot(\alpha-(k-1))}^{k}$$

$$= \frac{1}{k!}(-1)^k\,(-\alpha)(1-\alpha)\cdot\ldots\cdot(k-1-\alpha)$$

$$= \frac{(-1)^k}{k!}\,\underline{(k-\alpha-1)}\,(k-\alpha-2)\cdot\ldots\cdot(-\alpha)$$

$$= \frac{(-1)^k}{k!}\,(k-\alpha-1)^{\underline{k}} \qquad \binom{k-\alpha-1}{k}$$

$$\boxed{\binom{\alpha}{k} = (-1)^k \binom{k-\alpha-1}{k}}$$ górna negacja

$$\binom{-2}{k} = (-1)^k \binom{k+2-1}{k} = (-1)^k \binom{k+1}{k}$$

$$= (-1)^k \binom{k+1}{1} = (-1)^k (k+1).$$

Zadanie: wyznacz $\binom{-3}{k} = \ldots$

| | | | | | |
|---|---|---|---|---|---|
| $-2$: | 1 | $-2$ | 3 | $-4$ | 5 |
| $-1$: | 1 | $-1$ | 1 | $-1$ | 1 |
| 0: | 1 | 0 | 0 | 0 | 0 |
| 1: | 1 | 1 | 0 | 0 | 0 |
| 2: | 1 | 2 | 1 | 0 | 0 |
| 3: | 1 | 3 | 3 | 1 | 0 |
| 4: | 1 | 4 | 6 | 4 | 1 |

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$L(x) = \binom{x}{k} \in \mathbb{C}[x]$$

$\deg = k$

$$P(x) = \binom{x-1}{k} + \binom{x-1}{k-1} \in \mathbb{C}[x]$$

$\deg = k$

$\Leftarrow$ Ile jest $x$ t.że $L(x) = P(x)$ ?

Dla $\infty$ wielu $x$ (dla wszyst $n \in \mathbb{N}, n \geq k$)

Wniosek

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad k \geq 1$$