Współczynniki dwumianowe – III

$$\binom{-1/2}{k} = \frac{(-1/2)^{\underline{k}}}{k!} = \frac{1}{k!}\prod_{i=0}^{k-1}\left(-\frac{1}{2}-i\right) =$$

$$= \frac{1}{k!}(-1)^k \prod_{i=0}^{k-1}\left(i+\frac{1}{2}\right) = \frac{1}{k!}\left(\frac{1}{2}\right)^k \prod_{i=0}^{k-1}(2i+1)\cdot(-1)^k =$$

$$= \frac{(-1)^k}{k!}\cdot\frac{1}{2^k}\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1) =$$

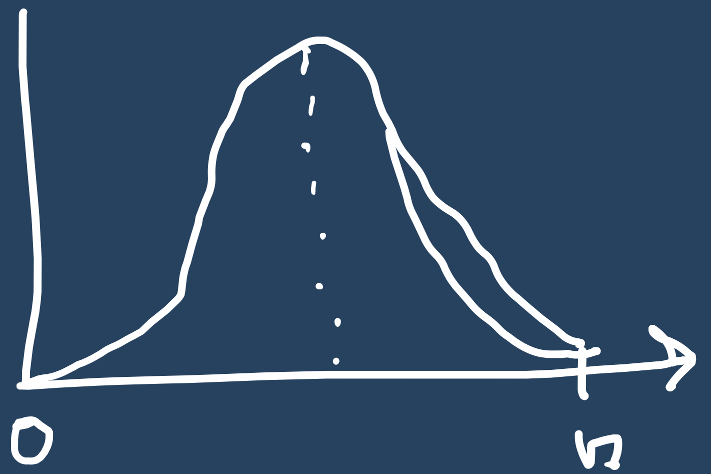
$$= \frac{(-1)^k}{k!}\,\frac{1}{2^k}\,\frac{1\cdot 2\cdot 3\cdot\ldots\cdot(2k-1)(2k)}{2\cdot 4\cdot 6\cdot\ldots\cdot(2k)} =$$

$$= \frac{(-1)^k}{k!}\,\frac{1}{2^k}\,\frac{(2k)!}{2^k\cdot 1\cdot 2\cdot\ldots\cdot k} = \frac{(-1)^k}{4^k}\,\frac{(2k)!}{k!\cdot k!}$$

$$\boxed{\binom{-1/2}{k} = (-1)^k\,\frac{1}{4^k}\binom{2k}{k}}$$

$$\left(\binom{n}{k}\right)_{k=0\ldots n} :$$

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} > 1$$



$$\binom{2k}{k} \sim \ldots$$

WZÓR STIRLINGA: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

równoważność asymptotyczna

czyli

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} = \frac{\sqrt{4\pi k}\left(\frac{2k}{e}\right)^{2k}}{2\pi k\left(\frac{k}{e}\right)^{2k}} = \sqrt{\frac{4\pi k}{4\pi^2 k^2}}\left(2\right)^{2k}$$

$$= \frac{1}{\sqrt{\pi k}} \cdot 4^k.$$

$$\binom{-1/2}{k} = (-1)^k \frac{1}{4^k}\binom{2k}{k} \sim (-1)^k \frac{1}{4^k}\cdot\frac{1}{\sqrt{\pi k}}\cdot 4^k$$

$$= (-1)^k \frac{1}{\sqrt{\pi k}}.$$

Jak obliczyć $\binom{1/2}{k}$?

WIEMY, ŻE : $(k \geq 1)$

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$ prawo absorpcji

$$\binom{z}{k} = \frac{z}{k}\binom{z-1}{k-1} \qquad z \in \mathbb{C}$$

$$\binom{1/2}{k} = \frac{1/2}{k}\binom{-1/2}{k-1} = \ldots \Leftarrow (Z)$$

Uogólniony wzór dwumianowy

$$f(x) = (1+x)^{\alpha} \qquad \alpha \in \mathbb{R}$$

$$f'(x) = \alpha(1+x)^{\alpha-1}$$

$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$$

$\vdots$

$$f^{(k)}(x) = \alpha^{\underline{k}}(1+x)^{\alpha-k}$$

// indukcja

$$f^{(0)}(x) = f(x)$$

wzór Taylora (dla $x=0$)

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_{n+1}(x)$$

$$\parallel$$

$$(1+x)^\alpha = \sum_{k=0}^{n} \frac{\alpha^{\underline{k}} \left.(1+x)^{\alpha-k}\right|_{x=0}}{k!} \cdot x^k + R_{n+1}(x)$$

$$= \sum_{k=0}^{n} \binom{\alpha}{k} x^k + R_{n+1}(x)$$

FAKT: $|x| < \frac{1}{2} \longrightarrow |R_{n+1}(x)| \xrightarrow[n\to\infty]{} 0$

$$h(x) = \sum_{n \geq 0}^{\infty} \binom{\alpha}{n} x^n$$: szereg potęgowy

ZADANIE: $R = 1$

$$\boxed{(1+x)^{\alpha} = \sum_{n \geq 0} \binom{\alpha}{n} x^n, \quad |x| < 1 \qquad \alpha \in \mathbb{R}}$$

β

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = \sum_{n\geq 0} \binom{-2}{n} (-x)^n =$$

$$= \sum_{n\geq 0} (-1)^n \binom{n-(-2)-1}{n} (-1)^n x^n$$

$$= \sum_{n\geq 0} \binom{n+1}{n} x^n = \sum_{n\geq 0} \binom{n+1}{1} x^n$$

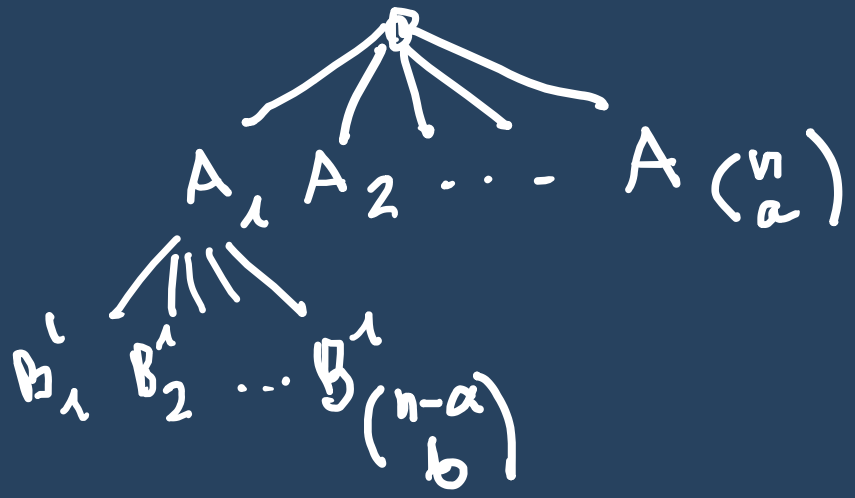
$$= \sum_{n\geq 0} (n+1) x^n \quad \square \qquad |x| < 1$$

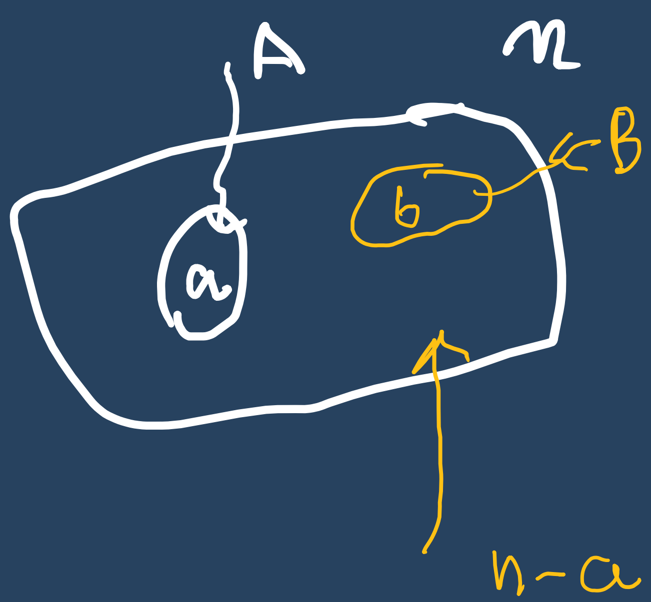
Współczynniki multimianowe.

▶ Ile jest par zbiorów (A, B) t.że

- $|A| = a$, $|B| = b$
- $A \cap B = \emptyset$
- A i $B \subseteq [n]$ $(= \{1, \dots, n\})$

Drzewo decyzyjne!





Odp: $\binom{n}{a}\binom{n-a}{b}$

$$\binom{n}{a}\binom{n-a}{b} = \frac{n!}{a!\,\cancel{(n-a)!}}\,\frac{\cancel{(n-a)!}}{b!\,(n-(a+b))!}$$

$$= \frac{n!}{a!\,b!\,(n-(a+b))!}$$

DEF: Jeśli

$a_1, \ldots, a_k \geq 0$, $a_1 + \ldots + a_k = n$

$$\binom{n}{a_1\ a_2\ \ldots\ a_k} = \frac{n!}{a_1!\,a_2!\ldots a_k!}$$

ODP :

$$\binom{n}{a \quad b \quad n-(a+b)}$$

ANALOGICZNIE :

$A_1, \ldots, A_k \subseteq [n]$, parami rozł.

$|A_i| = a_i$.

Ich mamy

$$\binom{n}{a_1 \quad a_2 \ldots a_k \quad n-(a_1+\ldots+a_k)}$$

To samo:

- $A_1, \dots, A_k \subseteq [n]$,
- $|A_i| = a_i$
- $A_1, \dots, A_k$ – parami rozł.
- $A_1 \cup \dots \cup A_k = [n]$

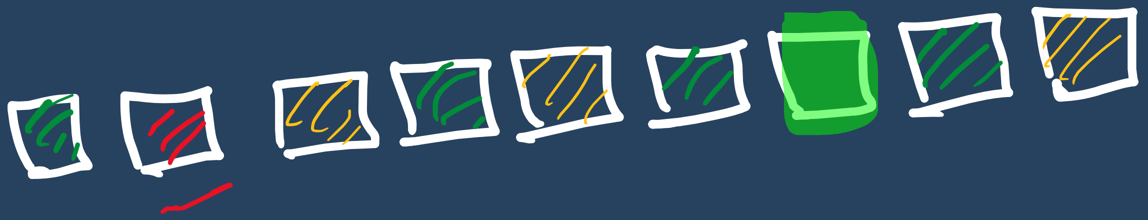
$\| \quad (A_1, \dots, A_k) \Leftarrow$ rozbicie $[n]$

Ilość rozbić: $\binom{n}{a_1\, a_2 \dots a_k}$

P Ile anagramów ma słowo

MISSISIPI ?

$n = 9$



$I \to 4$

$S \to 3$

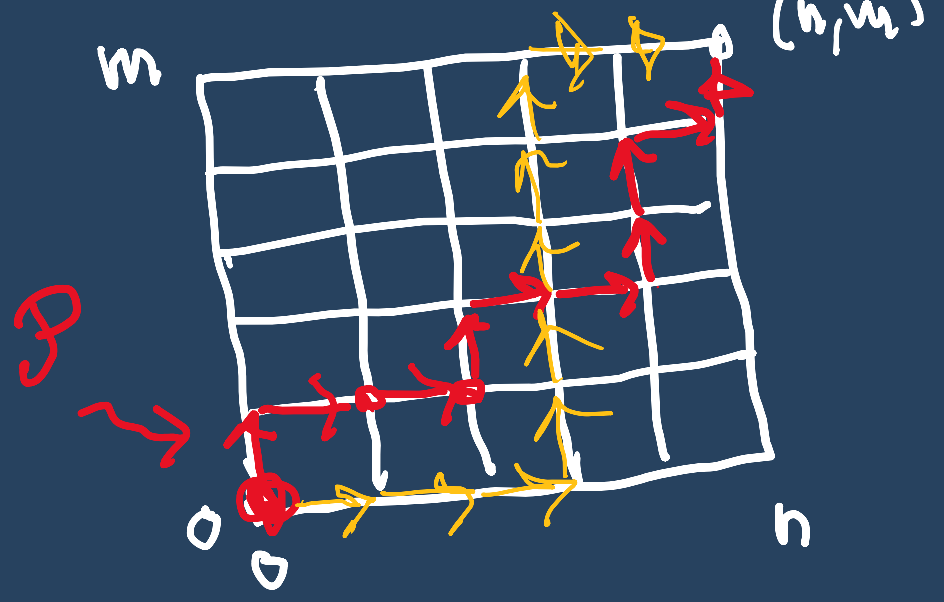
$M \to 1$

$P \to 1$

odp $\binom{9}{1\ 1\ 4\ 3} = \dfrac{9!}{1!\,1!\,4!\,3!}$

$= 2520$

SCIEZKI



Ile jest $\{\uparrow, \rightarrow\}$ sciezek ~~[illegible]~~ od $(0,0)$ do (n,m)?

$$P: \underbrace{\uparrow \rightarrow \rightarrow \uparrow \rightarrow \rightarrow \uparrow \uparrow \rightarrow \uparrow}_{10}$$

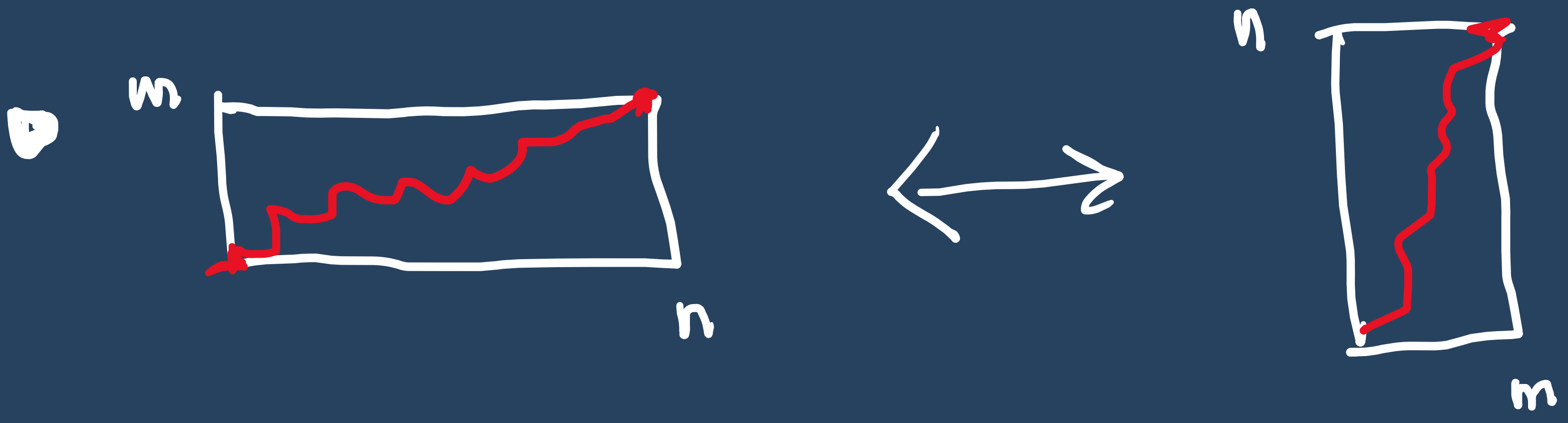
$\{\uparrow, \rightarrow\}$ - sciezki :

ciąg $((x_i, y_i))_{i=0 \ldots n+m}$

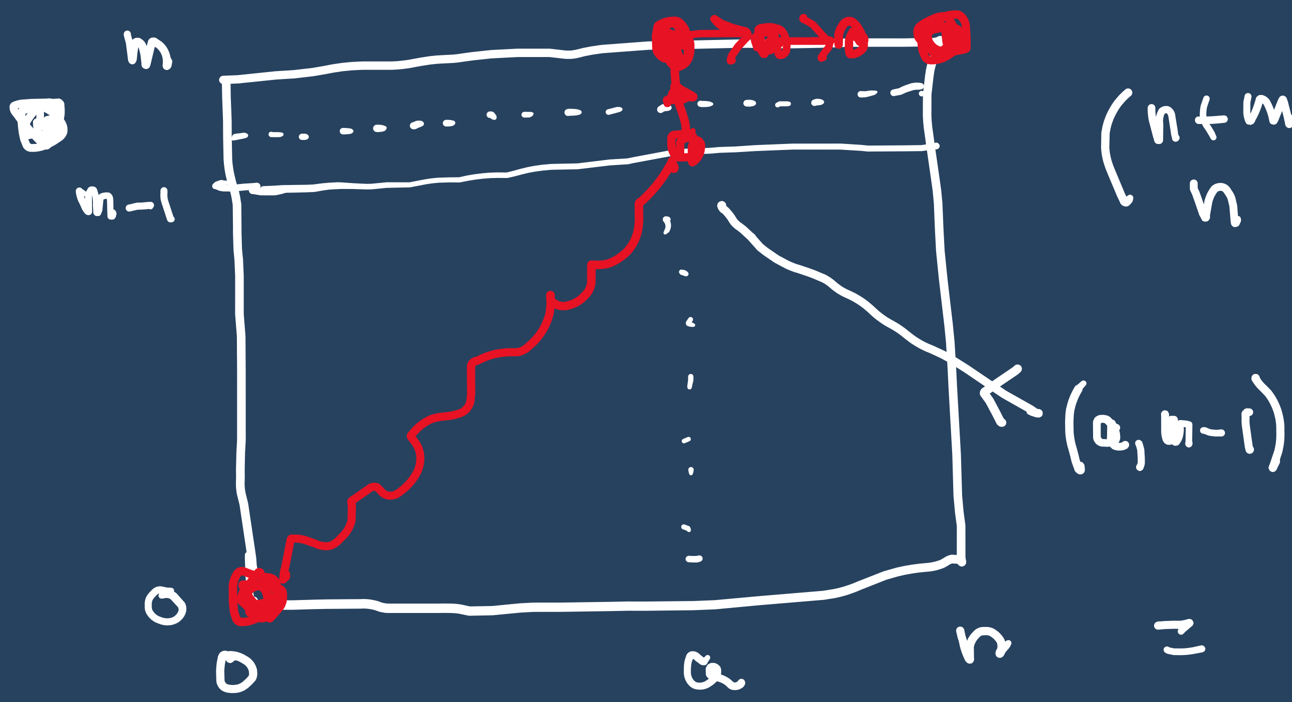
- $(x_0, y_0) = (0,0)$
- $(x_{n+m}, y_{n+m}) = (n,m)$
- $(x_{i+1}, y_{i+1}) = (x_i + 1, y_i) \vee$

  $(x_{i+1}, y_{i+1}) = (x_i, y_i + 1)$

TEGO MAMY $\binom{n+m}{n}$!!!



$$\binom{n+m}{n} = \binom{n+m}{m}$$

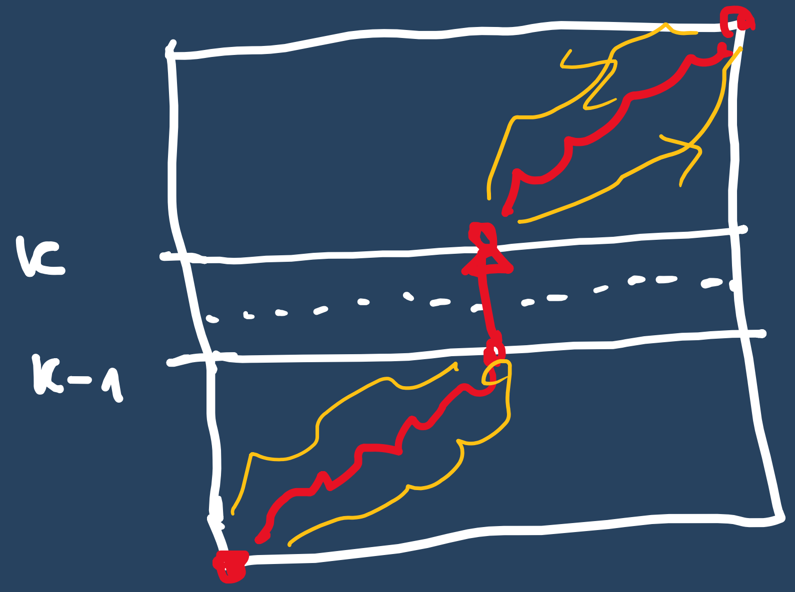


$$\binom{n+m}{n} = \sum_{a=0}^{n} \binom{a+m-1}{a} \cdot 1$$

$$= \sum_{a=0}^{n} \binom{m-1+a}{m-1} = \sum_{b \le n+m-1} \binom{b}{m-1}$$

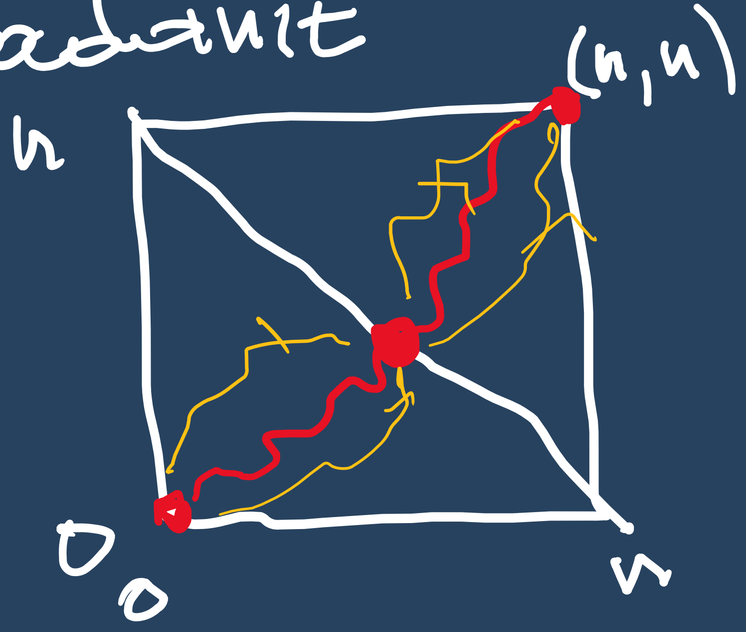
$$\sum_{b \le M} \binom{b}{c} = \binom{M+1}{c+1}$$

Zadanie



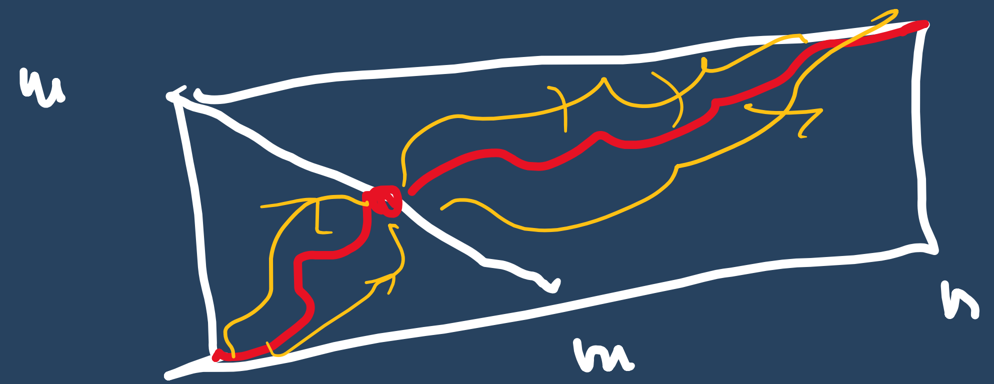
Jakie to jest zawarte prawo?

Zadanie



Jakie to jest prawo?

zadanie 1



$m \leq n$

$a \cdot (b \cdot (c \cdot d))$

$(a \cdot b) \cdot (c \cdot d)$

$\uparrow \uparrow \downarrow \downarrow$

$\uparrow \downarrow \uparrow \downarrow$