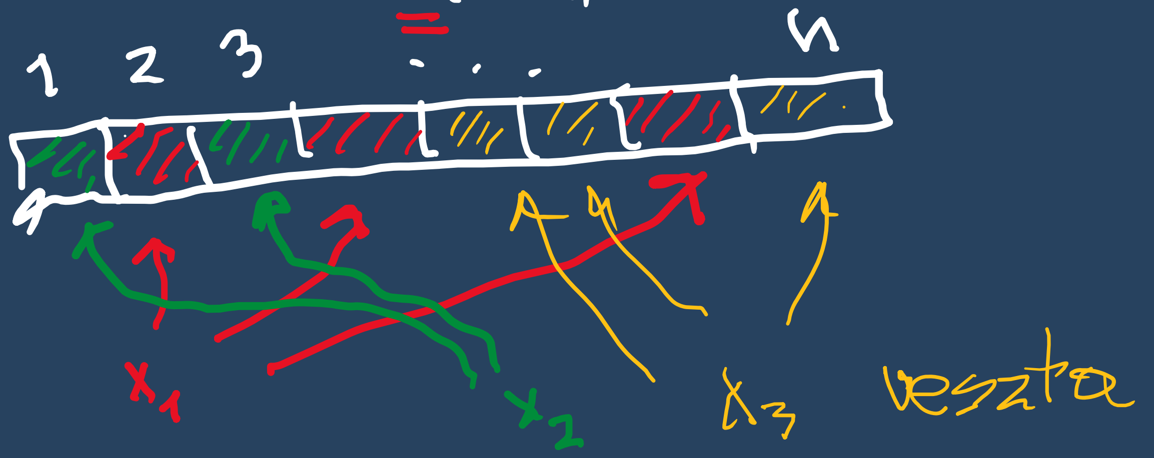
Wsp. multimianowe - cd.

$$(x_1+\dots+x_k)^n = (x_1+\dots+x_k)\cdot(x_1+\dots+x_k)\cdot\ldots\cdot(x_1+\dots+x_k)$$

$$= \sum_{\substack{a_1+\dots+a_k=n \\ a_1,\dots,a_k\geq 0}} C^n_{a_1,\dots,a_k}\, x_1^{a_1}\cdot\ldots\cdot x_k^{a_k}$$

$$(x_1+x_2+x_3)(x_1+x_2+x_3)\ldots(x_1+x_2+x_3)$$

$k=3$



$$\binom{n}{3,\,2,\,c} = C^n_{3,2,c}$$

$c = n-5$

$$(x_1 + \dots + x_k)^n = \sum_{\substack{a_1 + \dots + a_k = n \\ a_1, \dots, a_k \geq 0}} \binom{n}{a_1\ a_2 \dots a_k} x_1^{a_1} \cdot \dots \cdot x_k^{a_k}$$

UOGÓLNIONY WZÓR DWUMIANOWY

ZADANIE: zrób to ind. po n.

D) $k = 2$

$$(x_1 + x_2)^n = \sum_{\substack{a+b=n \\ a,b \geq 0}} \binom{n}{a\ \ b} x_1^a x_2^b = \sum_{a+b=n} \binom{n}{a} x_1^a x_2^b$$

$$\binom{n}{a\ \ b} = \frac{n!}{a!\,b!} = \frac{n!}{a!\,(n-a)!} = \binom{n}{a}$$
$a+b=n$

$$= \sum_{a=0}^{n} \binom{n}{a} x_1^a x_2^{n-a}.$$

P $\quad k^n = (1+\dots+1)^n = \sum_{\substack{a_1+\dots+a_k=n \\ a_1,\dots,a_k \geq 0}} \binom{n}{a_1\, a_2 \dots a_k}$

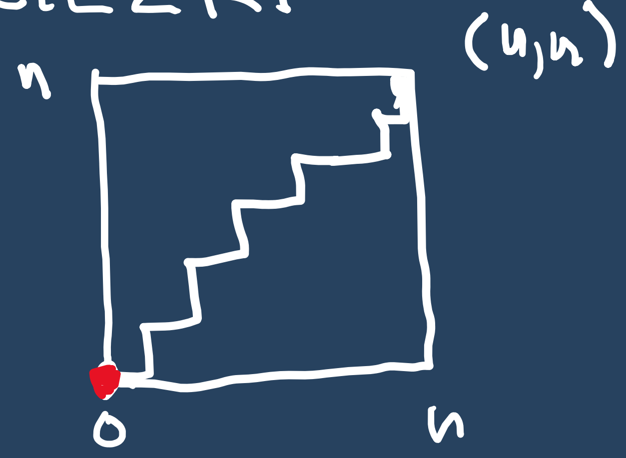
$$3^n = \sum_{a+b+c=n} \binom{n}{a\ b\ c}$$

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k\ \ n-k}$$

P $\quad 1^n = (1+1+(-1))^n =$

$$= \sum_{a+b+c=n} \binom{n}{a\ b\ c} (-1)^c .$$

Ścieżki



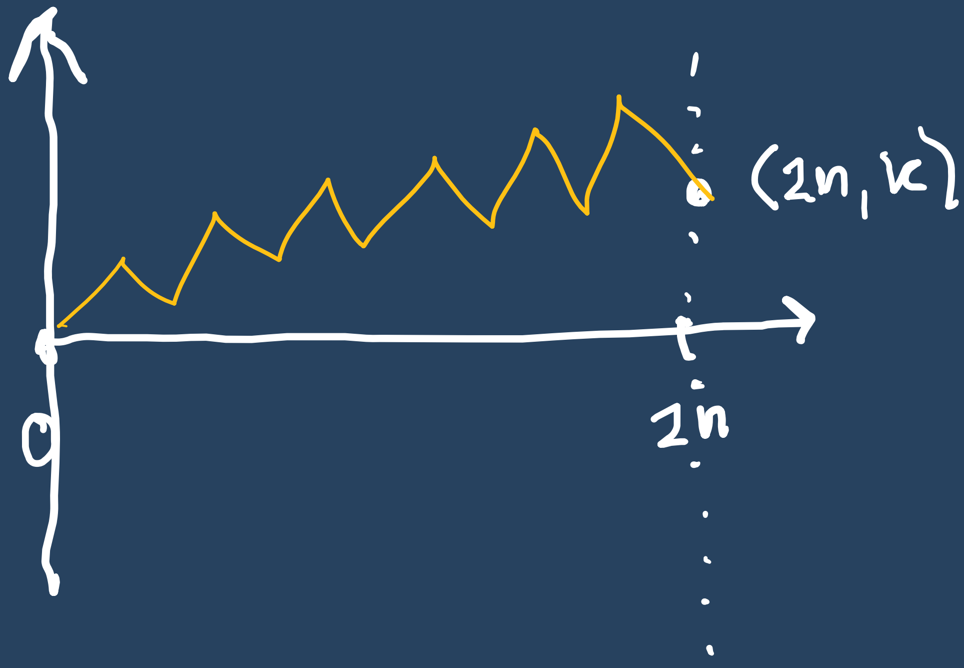
$\xrightarrow[O_{-45^\circ}]{}$

$\{\uparrow, \rightarrow\}$ - ścieżki $\rightsquigarrow$ $\{\nearrow, \searrow\}$ - ścieżki

$\uparrow$

ciąg $\{(k, y_k)\}_{k=0..2n}$ $: |y_{k+1} - y_k| = 1$

$y_0 = 0$



Ile jest ścieżek od $(0,0)$ do $(2n,k)$

u = liczba $\nearrow$

d = liczba $\searrow$

$$\begin{cases} u + d = 2n \\ u - d = k \end{cases} \quad \downarrow (+)$$

$$2u = 2n + k$$

Zatem $2 \mid k$.

$$\begin{cases} u + d = 2n \\ u - d = 2k \end{cases}$$

$$2u = 2n + 2k$$

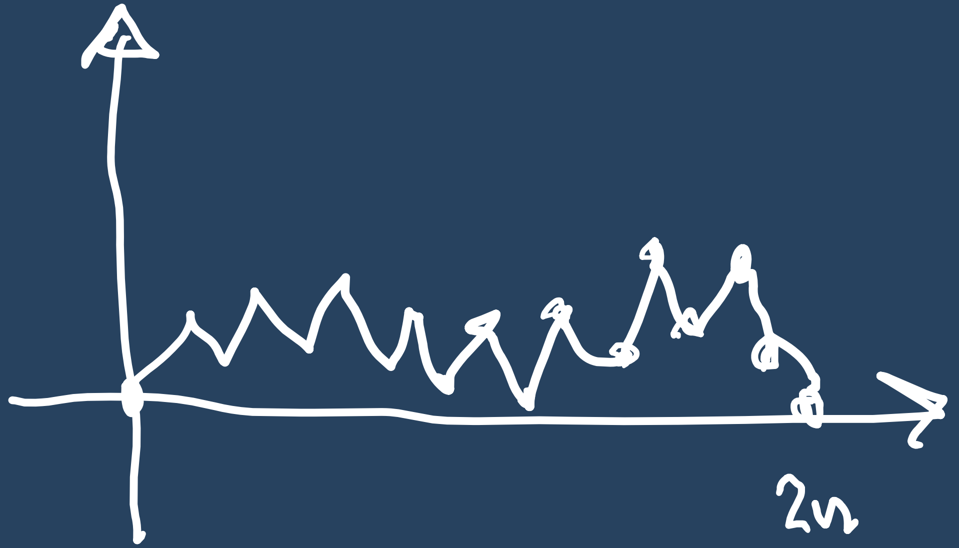
$$\begin{cases} u = n + k \\ d = 2n - (n+k) = n - k \end{cases}$$

wn. Liczba $\{\nearrow, \searrow\}$ - ścieżek od $(0,0)$ do $(2n, 2k)$ $= \binom{2n}{n-k}$ $\left(= \binom{2n}{n+k}\right)$

Def. Ścieżka Dyke'a:

taka $\{\nearrow, \searrow\}$ - ścieżka $\{(k, y_k)\}_{k=0\ldots 2n}$

t. że $y_0 = 0$, $y_{2n} = 0$ oraz $(\forall k)(y_k \geq 0)$



D_n = liczba ścieżek Dycka długości $2n$.

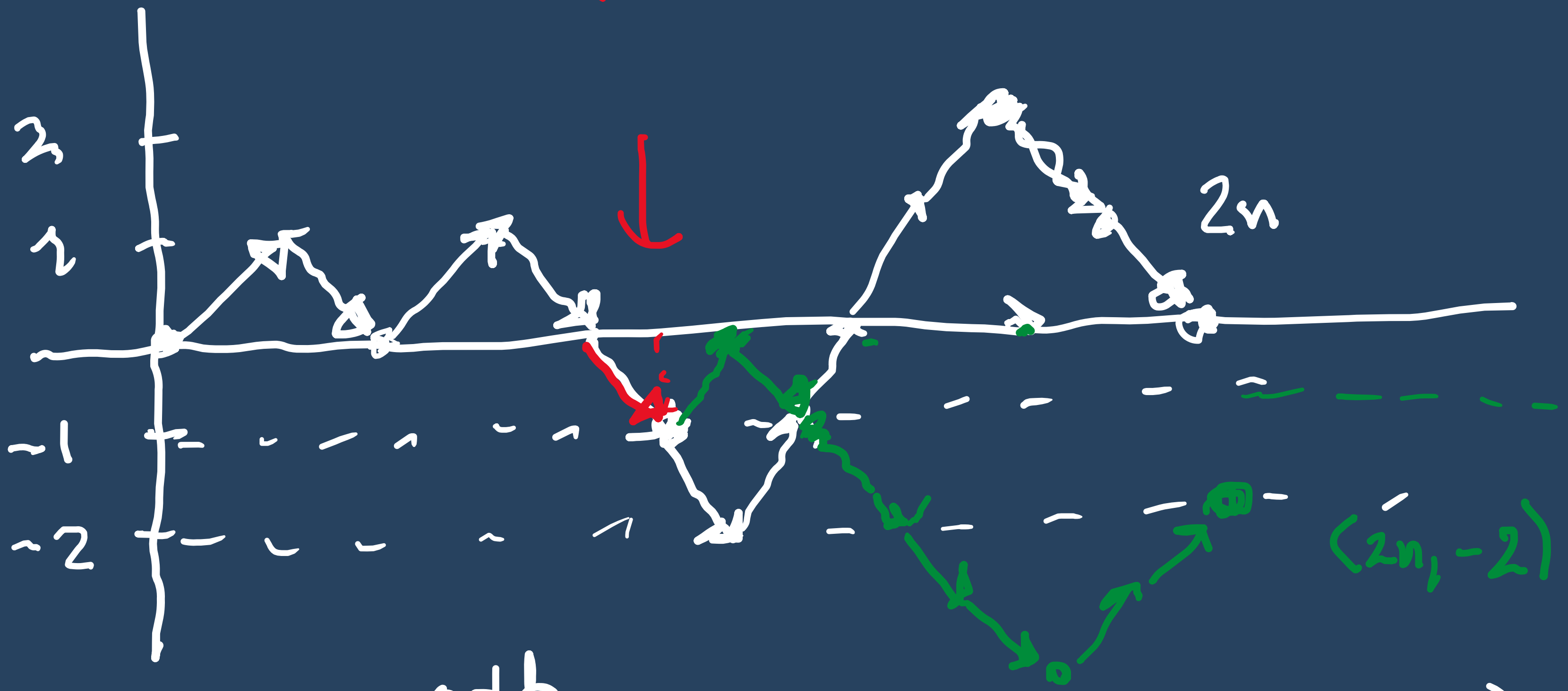
D_n = liczba wszystkich ścieżek do $(2n, 0)$

− liczba złych ścieżek

$= \binom{2n}{n}$ − liczba złych.

złe ścieżki

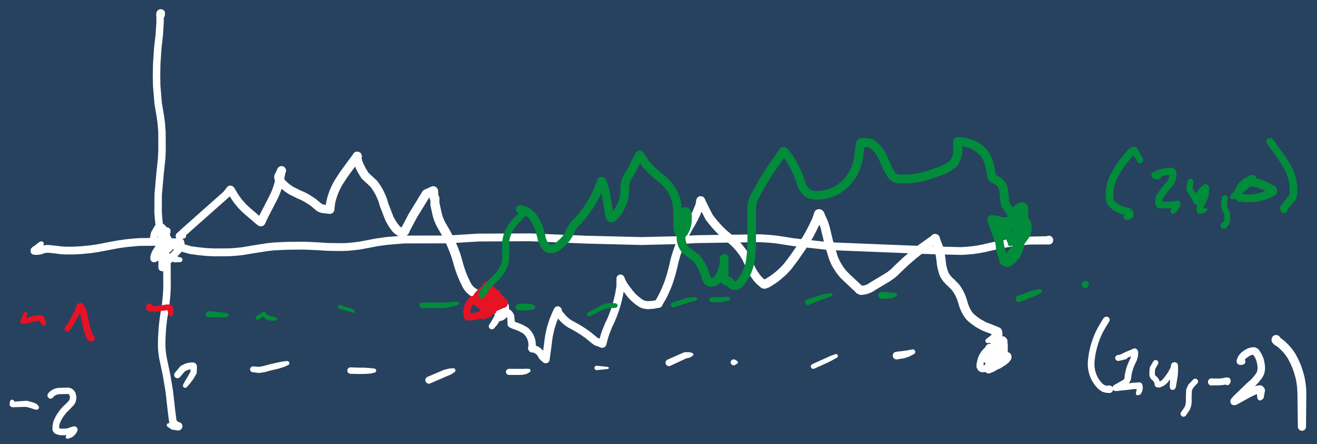
pierwsze złe miejsce



odbicie wzgl. $\mathbb{N} \times \{-1\}$

$$\text{zła} \xrightarrow{\text{odb.}} \text{ścieżka do } (2n, -2)$$

bijekcja



liczba złych = liczba ścieżek do $(2n,-2)$

$$= \binom{2n}{n-1}$$

$$D_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)!}$$

$$= \frac{(2n)!}{n!\,n!}\left(1 - \frac{n}{n+1}\right)$$

$$= \binom{2n}{n}\left(\frac{n+1-n}{n+1}\right)$$

$$= \frac{1}{n+1}\binom{2n}{n}.$$

$$\frac{1}{(n-1)!} = \frac{n}{n!}$$

DEF. n-ta liczba Catalana:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

wn. $C_n = \binom{2n}{n} - \binom{2n}{n+1}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
. . . .

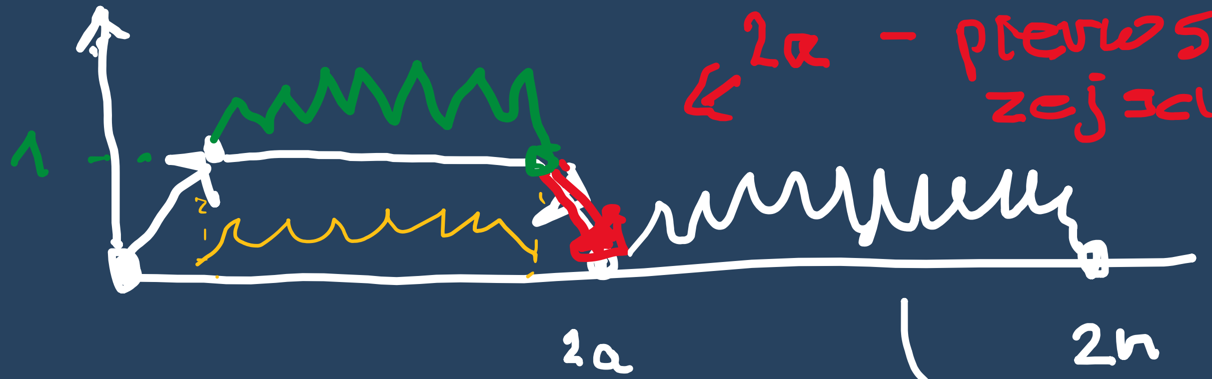
$$C_3 = \frac{1}{4}\binom{6}{3} = \frac{20}{4} = 5$$

$C_0 = 1$

$C_1 = 1$ $\qquad C_2 = 2$

Wn. $\boxed{D_n = C_n}$

WZÓR REKURENCYJNY



2a – pierwsze zejście do "0"

ścieżki Dyck'a

ścieżka 2(n-a)

: ścieżka Dyck'a dług. 2(a-1)

+ (-1, -1)

$$C_n = \sum_{a=1}^{n} C_{a-1} \cdot C_{n-a}$$

$$C_n = \sum_{a=1}^{n} C_{a-1} \cdot C_{n-a}$$

$$// \; i = a-1$$

$$a = i+1$$

$$n-a = n-(i+1)$$
$$= (n-1)-i$$

$$= \sum_{i=0}^{n-1} C_i \cdot C_{(n-1)-i}$$

czyli:

$$C_{n+1} = \sum_{i=0}^{n} C_i \cdot C_{n-i}$$

P $C_4 = C_0 \cdot C_3 + C_1 \cdot C_2 + C_2 \cdot C_1 + C_3 \cdot C_0$

$= 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1$

$= 14.$

Z obliczanie C_5,