Metoda sum Riemanowskich

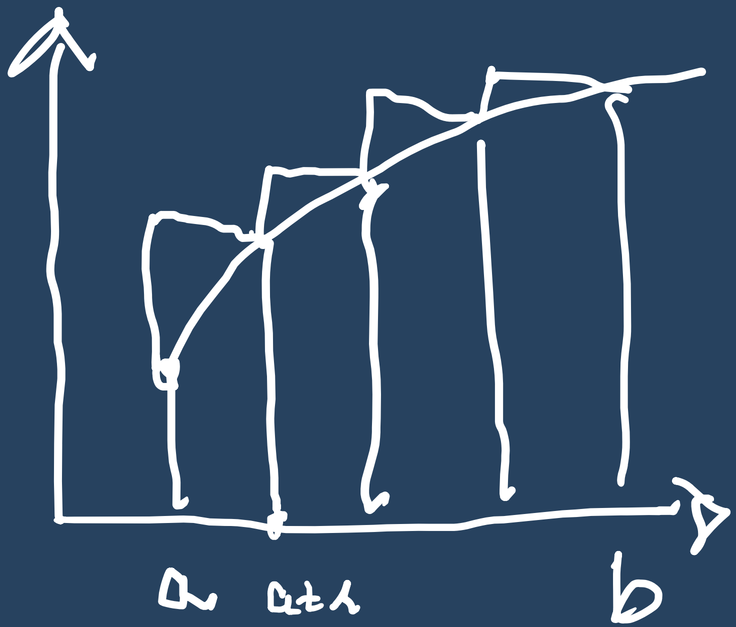
$f: [a, b] \longrightarrow \mathbb{R}$, $a, b \in \mathbb{N}$

$a \leq x < y \leq b \rightarrow f(x) \leq f(y)$

CEL: oszacowanie $\sum_{k=a}^{b} f(k)$.

$$\sum_{k=a}^{b-1} f(k) \leq \int_a^b f(x)\,dx$$

$$\sum_{k=a}^{b} f(k) \leq \int_a^b f + f(b)$$



$$\int_a^b f\,dx \le \sum_{k=a+1}^{b} f(k)$$

$$f(a)+\int_a^b f(x)\,dx \le \sum_{k=a}^{b} f(k)$$

ZATEM:

$$f(a)+\int_a^b f(x)\,dx \le \sum_{k=a}^{b} f(k) \le \int_a^b f(x)\,dx + f(b)$$

P $\quad \sum_{k=1}^{n} k^2 \quad \left(= \frac{1}{6} n(n+1)(2n+1) \right)$

$$0 + \int_0^n x^2\,dx \le \sum_{k=1}^{n} k^3 \le \int_1^n x^2\,dx + n^2$$

$$\frac{1}{3} n^3 \qquad\qquad = \frac{1}{3} n^3 + n^2$$

$$\frac{1}{3} n^3 \le \sum_{k=1}^{n} k^2 \le \frac{1}{3} n^3 + n^2$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{3} n^3 + O(n^2)$$

Ⓟ $n! = 1\cdot 2\cdot 3\cdot\ldots\cdot n$

$$\ln(n!) = \sum_{k=1}^{n} \ln(k) \qquad f(x) = \ln(x)$$

$$\int_1^n \ln(x)\,dx = \int_1^n x'\ln(x)\,dx = x\ln(x)\Big|_1^n - \int x\ln'(x)\,dx$$

$$= n\ln(n) - \int_1^n 1\,dx = n\ln(n) - (n-1)$$

$$= n(\ln(n)-1)+1 = n\ln\left(\frac{n}{e}\right)+1 =$$

$1=\ln(e)$

$$= \ln\left(\frac{n}{e}\right)^n + 1$$

$$\ln\left(\frac{n}{e}\right)^n + 1 \leq \ln(n!) \leq \ln\left(\frac{n}{e}\right)^n + 1 + \ln(n)$$

$$\ln\left(\left(\frac{n}{e}\right)^n \cdot e\right) \leq \ln(n!) \leq \ln\left(\left(\frac{n}{e}\right)^n \cdot e \cdot n\right)$$

$$e \cdot \left(\frac{n}{e}\right)^n \leq n! \leq \left(\frac{n}{e}\right)^n \cdot e \cdot n$$

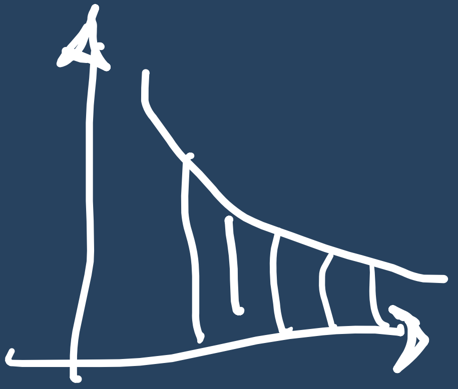
Stirling:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

A skąd π w : później

ANALOGICZNIE

$f: [a,b] \longrightarrow \mathbb{N}$; $a \le x < y \le b \to f(x) \geq f(y)$



$$\int_a^b f + f(b) \le \sum_{k=a}^{b} f(k) \le \int_a^b f + f(1)$$

ZADANIE :

Pokaż to (lista zadań)

(P) **Def.** n-ta liczba harmoniczna

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \qquad f(x) = \frac{1}{x}$$

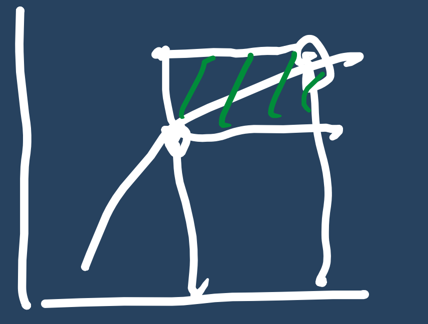
$$\ln(n) \leq \frac{1}{n} + \ln(n) \leq H_n \leq \int_1^n \frac{1}{x}\,dx + 1 = \ln(n) + 1$$

$$\ln(n) \leq H_n \leq \ln(n) + 1$$

$$H_n = \ln(n) + \alpha_n \qquad 0 \leq \alpha_n < 1$$

Dokładniej.

$$(+)\downarrow \begin{cases} \ln(n) - \ln(n-1) = a_n \\ \ln(n-1) - \ln(n-2) = a_{n-1} \\ \vdots \\ \ln(2) - \ln(1) = a_2 \end{cases}$$



suma teleskopowa

$$\ln(n) = \sum_{k=2}^{n} a_n = \sum_{k=2}^{n} \left(\ln(k) - \ln(k-1)\right)$$

$$= \sum_{k=2}^{n} \ln \frac{k}{k-1} = \sum_{k=2}^{n} \ln \frac{1}{1-\frac{1}{k}} .$$

$$\ln \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{1}{k} x^k \qquad |x|<1$$

$$= \frac{1}{1}\cdot x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

Zadanie: Udowodnij to

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \qquad |x|<1$$

scałkuj obie strony

$$\ln(n) = \sum_{k=2}^{n} \ln \frac{1}{1-\frac{1}{k}} = \sum_{k=2}^{n} \sum_{a=1}^{\infty} \frac{1}{a}\left(\frac{1}{k}\right)^{a}$$

$$= \sum_{a=1}^{\infty} \frac{1}{a} \sum_{k=2}^{n} \left(\frac{1}{k}\right)^{a} =$$

$$= \left(\sum_{k=2}^{n} \frac{1}{k} + \underbrace{\sum_{a \geq 2} \frac{1}{a} \sum_{k=2}^{n} \left(\frac{1}{k}\right)^{a}}_{(*)}\right.$$

$$= H_n - 1 + (*)$$

$$H_n = \ln(n) + 1 - \sum_{a \geq 2} \frac{1}{a} \underbrace{\sum_{k=2}^{n} \left(\frac{1}{k}\right)^a}_{\text{rośnie ze wzrostem } n}$$

$H_n - (\ln n + 1)$ → malejący

← zbieżne!!

$$1 - \gamma = \lim_{n \to \infty} H_n - (\ln(n) + 1)$$

$$H_n = \ln n + \gamma + o(1)$$

$\gamma = 0.577\ldots$ $\leftarrow$ stała Eulera

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120 \cdot n^4} - \ldots$$

$$H_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

MULTIZBIORY

idea: zbiory z powtórzeniami

(bazy danych, Big Data)

Ogólna definicja:

ustalmy Ω

rozważamy $f \in \mathbb{N}^{\Omega}$.

idea: $f(\omega)$ = częstość wyst. ω

$$\ker(f) = \{\omega \in \Omega : f(\omega) > 0\}$$

$$\mathrm{MULT}(\Omega) = \{f \in \mathbb{N}^{\Omega} : |\ker(f)| < \infty\}.$$

$$f \in \mathrm{MULT}(\Omega) : |f| = \sum_{\omega \in \Omega} f(\omega).$$

- $f, g \in \mathcal{MULT}(\Omega)$
  - $(f+g)(\omega) = f(\omega) + g(\omega)$
  - $(f \vee g)(\omega) = \max(f(\omega), g(\omega))$

SZCZEGÓLNY PRZYPADEK: $|\Omega| < \infty$.

$$\mathcal{MULT}(\Omega) = \mathbb{N}^{\Omega}.$$

OZNACZENIE: $\mathcal{MULT}(n) = \mathcal{MULT}(\{1, \dots, n\})$

Pytanie: mamy n, k ustalone

$$\left|\{ f \in \mathcal{M}ult(n) : |f| = k \}\right| = ?$$

P $n = 3$: $\Omega = \{1, 2, 3\}$

$f \in \mathcal{M}ULT(\Omega)$; $f : \{1, 2, 3\} \longrightarrow \mathbb{N}$

$f(1) = 2$; $f(2) = 3$; $f(3) = 1$

$f \leftrightarrow 2 \cdot 1 + 3 \cdot 2 + 1 \cdot 3$

$\Omega = \{a, b, c\}$

$$3 \cdot a + 2 \cdot b + 1 c \longleftrightarrow a^3 b^2 c^1$$

$$\longleftrightarrow aaabbc$$

$\Omega = \{1, 2, 3, 4\}$

$|f| = K \quad ; \quad f \in \mathcal{M}(\Omega)$

$$a \cdot 1 + b \cdot 2 + c \cdot 3 + d \cdot 4$$

- $a, b, c, d \in \mathbb{N}$
- $a + b + c + d = K$

$k = 7 \qquad a + b + c + d = 7$

$1 + 2 + 1 + 3 = 7$

$\underbrace{1}_{1} \; \underbrace{2\;2}_{2} \; \underbrace{3}_{1} \; \underbrace{4\;4\;4}_{3}$

0 | 0 0 | 0 | 0 0 0 rozmiar

ile to jest symboli : $k + (n-1)$

liczba przegródek

P 0 | | 0 0 | 0 0 0 | | 0

$1 \cdot 1 + 0 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 0 \cdot 5 + 1 \cdot 6$

liczba rozw. równ. $a_1 + a_2 + \dots + a_n = k$

$$0|0|00|0|00|000|0| \leftarrow k+(n-1)$$

kreski $n-1$

$$\binom{k+(n-1)}{n-1}$$

DEF $\boxed{\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}}$

liczby multizbiorowe

WNIOSEK:

$$\left|\{f \in \mathcal{M}(n) : |f| = k\}\right| = \left(\!\binom{n}{k}\!\right)$$

$$= \binom{n+k-1}{k}.$$

$\left(\!\binom{n}{k}\!\right) = \Big\{$ liczba rozw. równania

$x_1 + \ldots + x_n = k$

w $\mathbb{N}$

Obserwacja:

$$\binom{-n}{k} = (-1)^k \binom{k-(-n)-1}{k} =$$

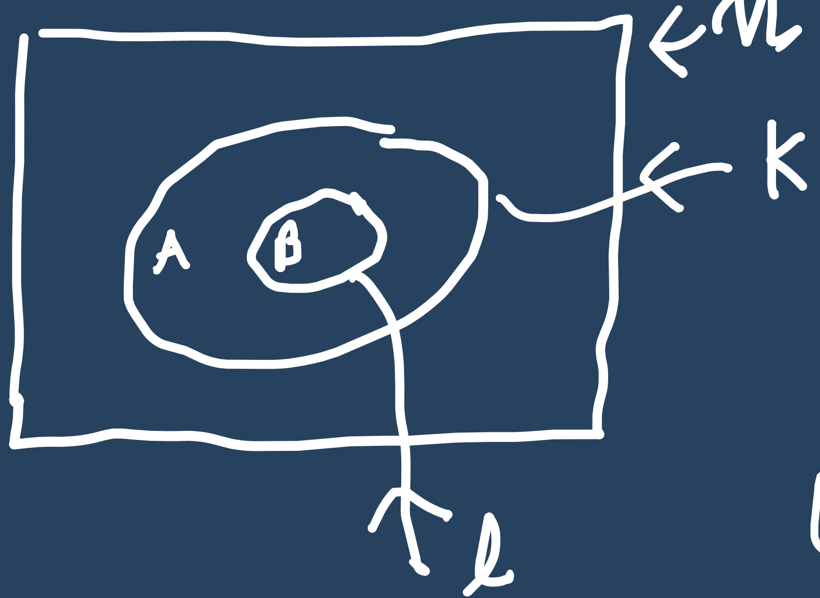
$$= (-1)^k \binom{k+n-1}{k} =$$

$$= (-1)^k \left(\!\binom{n}{k}\!\right)$$

$$\left(\!\binom{n}{k}\!\right) = (-1)^k \binom{-n}{k}$$

JESZCZE JEDNA TOŻSAMOŚĆ:

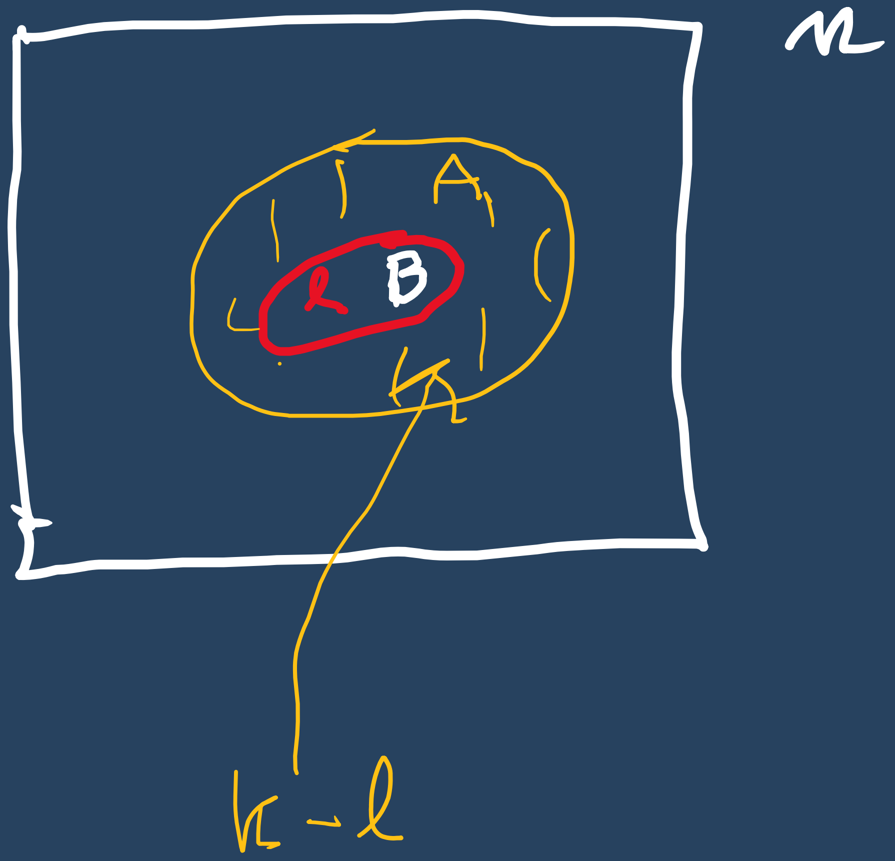
$$\binom{n}{k}\binom{k}{\ell} = (*)$$



1) wybieramy $A \in [n]^k$

2) wybieramy $B \in [A]^\ell$

$$(*) = \left| \{ (A,B) : A \in [n]^k \wedge B \in [k]^\ell \} \right|$$

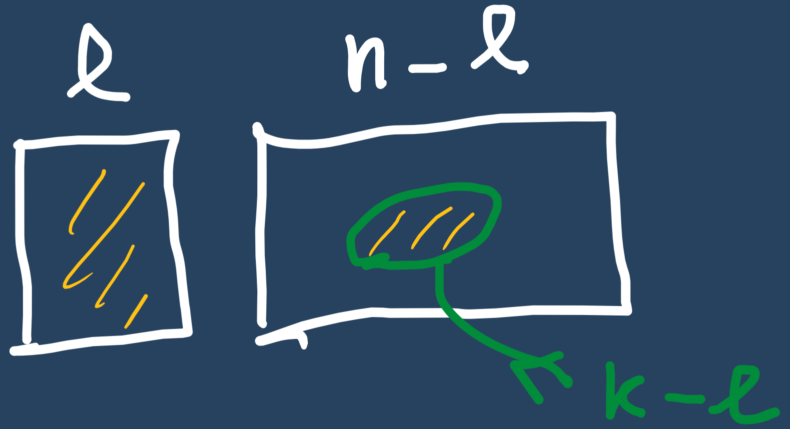


$$\binom{n}{k}\binom{k}{l}$$

$$\binom{n}{l} \cdot \square$$

1) $B \subseteq A$
2) $|A| = k$

$$\left.\begin{array}{l} B \subseteq A \\ |A| = k \end{array}\right\} \to |A \setminus B| = k - l$$



$$l + (k - l) = K$$

Tw. Jeśli $0 \leq l \leq k \leq n$, to

$$\boxed{\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}}$$