# Elementy rachunku prawdopodobieństwa.

Def. Kombinatoryczną przestrzenią probab. nazywamy parę $(\Omega, Pr)$, gdzie:

1) $\Omega \neq \emptyset$, $|\Omega| < \infty$

2) $Pr: P(\Omega) \longrightarrow [0,1]$ t. że

$$Pr(A) = \frac{|A|}{|\Omega|}.$$

Terminologia:

- elementy $\Omega$ : zdarzenia elementarne
- podzbiory $\Omega$ : zdarzenia

Podst. własności

- $0 \leqslant Pr(A) \leqslant 1$
- $Pr(\emptyset) = 0$

(3) • $Pr(\Omega) = 1$

- $A \subseteq B \to Pr(A) \leqslant Pr(B)$

(4) • $\left.\begin{array}{l} A \cap B = \emptyset \\ A, B \subseteq \Omega \end{array}\right] \to Pr(A \cup B) = Pr(A) + Pr(B)$

$$Pr(A) = \frac{|A|}{|\Omega|}$$

$$A \subseteq \Omega$$

Wnioski:

- $Pr(A^c) = 1 - Pr(A)$

// (3) & (4)

$\Omega = A \cup A^c$

$1 = Pr(A) + Pr(A^c)$

2 • $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

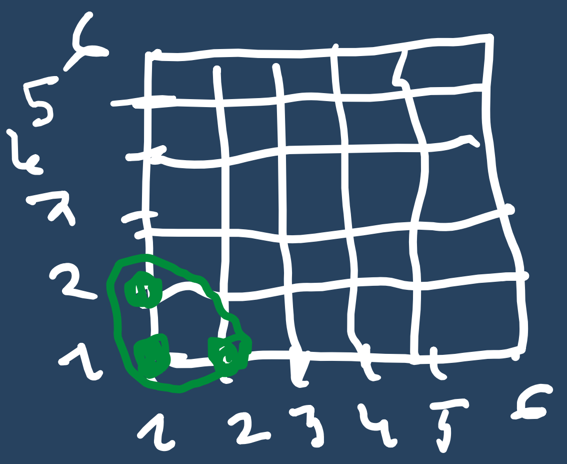
$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$

(P) $\Omega = \{1,\dots,6\} \times \{1,\dots,6\}$. ; $|\Omega| = 36$

- model 2-ch rzutów niesfałszowanymi kostkami do gry

$$Pr(\{(a,b)\}) = \frac{|\{(a,b)\}|}{36} = \frac{1}{36}.$$

- jakie jest prawd. wyrzucenia sumy $> 3$.




$$A = \{(a,b) \in \Omega : a+b > 3\}$$

$$A^c = \{(a,b) \in \Omega : a+b \leq 3\} =$$

$$= \{(1,1), (2,1), (1,2)\}$$

$$Pr(A^c) = \frac{3}{36} = \frac{1}{12} \; ; \; Pr(A) = 1 - \frac{1}{12} = \frac{11}{12}.$$

**Def.** 1) Zmienną losową na k.p.p. $(\Omega, Pr)$ nazywamy dowolną funkcję

$$X : \Omega \longrightarrow \mathbb{R}.$$

2) Wartością oczekiwaną zmiennej $X$ nazywamy liczbę

$$E(X) = \frac{\sum_{\omega \in \Omega} X(\omega)}{|\Omega|}$$

interpretacja:

"średnia wartość $X$"

Własności:

$$E(X) = \frac{\sum_{\omega} X(\omega)}{|\Omega|}$$

1) $X \equiv c \longrightarrow E(X) = c$

2) $E(a \cdot X) = a \cdot E(X)$

$(a \in \mathbb{R})$

3) $E(X+Y) = E(X) + E(Y)$ !!!

P $\Omega = \{1, \dots, 6\}^2$; $\quad S((a,b)) = a+b$.

$E[S] = ?$


$X((a,b)) = a$

$Y((a,b)) = b$

$S = X + Y \qquad E[S] = E[X] + E[Y]$

FAKT. Niech $X : \Omega \longrightarrow \mathbb{R}$.

Niech $A = \mathrm{rng}(X)$. Wtedy

$$E(X) = \sum_{a \in A} a \cdot \Pr\left[\{\omega \in \Omega : X(\omega) = a\}\right]$$

$$= \sum_{a \in A} a \cdot \Pr(X = a)$$

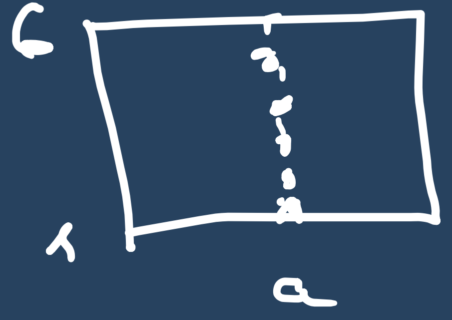
$/|\Omega|$

D-d

$$|\Omega| \cdot E(X) = \sum_{\omega} X(\omega) = \sum_{a \in A} \sum_{\substack{\omega \in \Omega \\ X(\omega) = a}} X(\omega) = \sum_{a \in A} a \cdot \sum_{\substack{\omega: \\ X(\omega) = a}} 1$$

Przykład c.d.

$X((a,b)) = a$




$$E(X) = \sum_{a=1}^{6} a \Pr(X=a)$$

$$= \sum_{a=1}^{6} a \cdot \frac{6}{36} = \frac{1}{6}\sum_{a=1}^{6} a$$

$$= \frac{1}{6}\,\frac{6\cdot 7}{2} = \frac{7}{2} \quad \left(= 3\tfrac{1}{2}\right)$$

$$E(S) = E(X) + E(Y) = 2\cdot\frac{7}{2} = 7.$$

(P) $\Omega = P(\{1, .., n\}) \qquad |\Omega| = 2^n.$

- ustalony $k$:

$$Pr(\{A \subseteq \{1, .., n\} : |A| = k\}) = \frac{\binom{n}{k}}{2^n}.$$

- $X(A) = |A|$

$$E(X) = \sum_{k=0}^{n} k \cdot Pr(X=k) = \sum_{k=0}^{n} k \frac{\binom{n}{k}}{2^n} =$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} k \cdot \binom{n}{k} = \frac{1}{2^n} \sum_{k=1}^{n} k \binom{n}{k} = \frac{1}{2^n} \sum_{k=1}^{n} \cancel{k} \cdot \frac{n}{\cancel{k}} \binom{n-1}{k-1}$$

$$= \frac{n}{2^n} \sum_{k=1}^{n} \binom{n-1}{k-1} = \frac{n}{2^n} \sum_{l=0}^{n-1} \binom{n-1}{l} = \frac{n}{2^n} \cdot 2^{n-1} = \frac{n}{2} \qquad \blacksquare$$

# Permutacje

rozkład na cykle

$S_n$ = zbiór permutacji $\{1, \ldots, n\}$

$|S_n| = n!$

P $\quad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 7 & 2 & 1 & 10 & 8 & 3 & 4 & 6 & 9 \end{pmatrix}$

$$\boxed{4 \to 1 \to 5 \to 10 \to 9 \to 6 \to 8 \to 4} \qquad \boxed{2 \to 7 \to 3 \to 2}$$

$$\pi = (1, 5, 10, 9, 6, 8, 4) \circ (2, 7, 3)$$

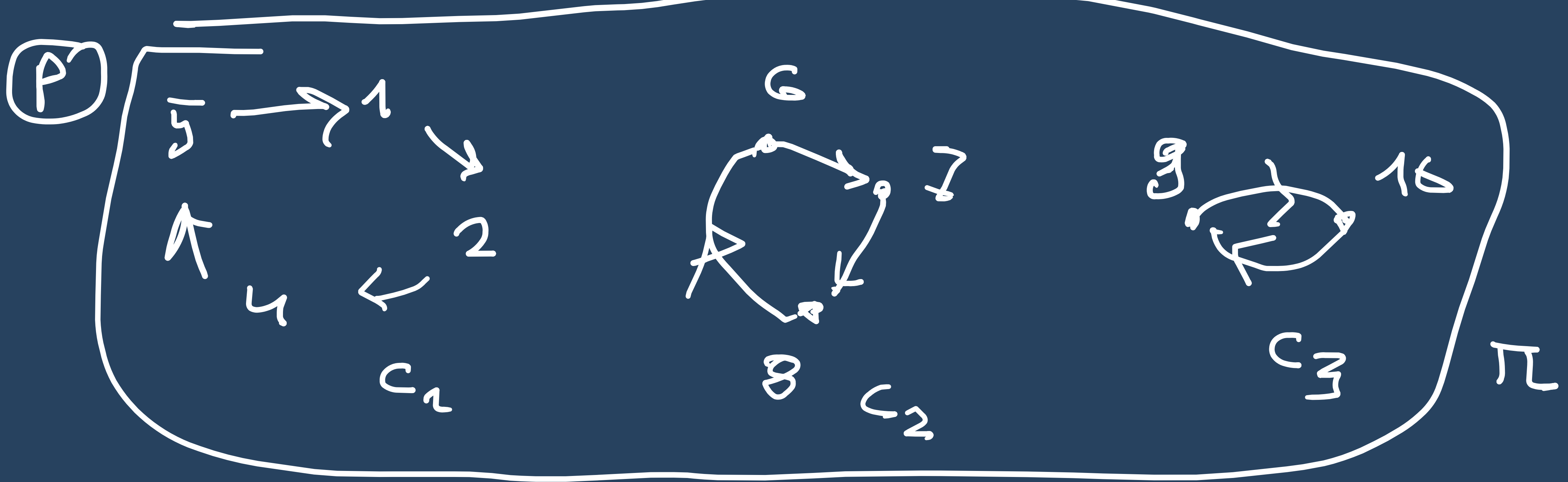
CYKLE

FAKT: Każda perm. można rozłożyć na rozłączne cykle

D-d Ustalmy $\pi \in S_n$. Def.

$$a \sim b \equiv (\exists k \in \mathbb{Z})(\pi^k(a) = b)$$

- $\sim$ – rel. równow. na $\{1, \ldots, n\}$
- CYKLE $\simeq \{1, \ldots, n\}/\sim$.




$$\operatorname{sgn}(\pi) = \operatorname{sgn}(C_1)\cdot\operatorname{sgn}(C_2)\cdot\operatorname{sgn}(C_3) = (-1)\cdot(1)\cdot(-1) = 1.$$

FAKT: $C$ - cykl. dług $k \rightarrow \operatorname{sgn}(C) = (-1)^{k+1}$

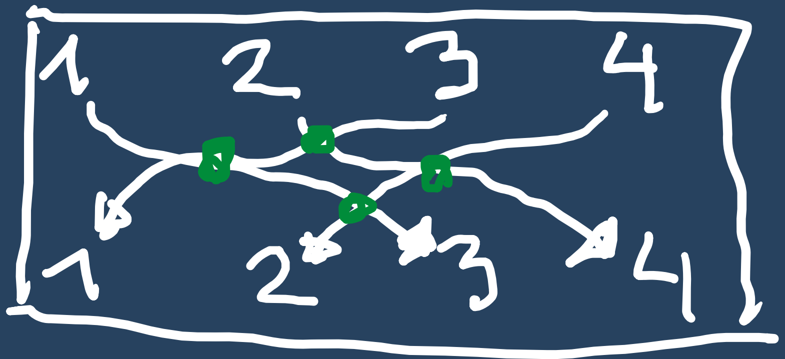
Wn. $\pi = C_1 \circ C_2 \circ \ldots \circ C_k$, $|C_i| = c_i$. $\pi \in S_n$

$$\mathrm{sgn}(\pi) = (-1)^{c_1+1} \cdot \ldots \cdot (-1)^{c_k+1} =$$

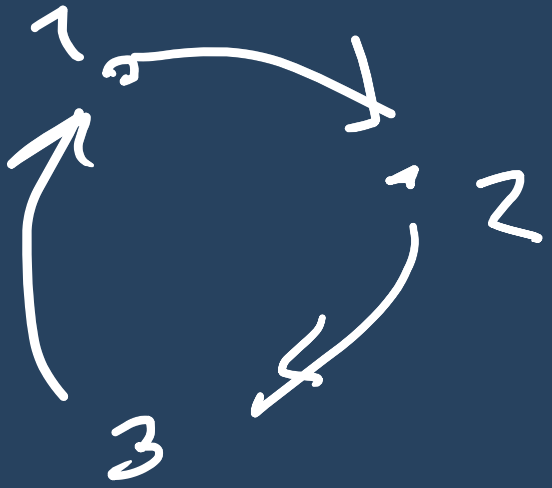
$$= (-1)^{(c_1+\ldots+c_k)+k} = (-1)^{n+k}.$$

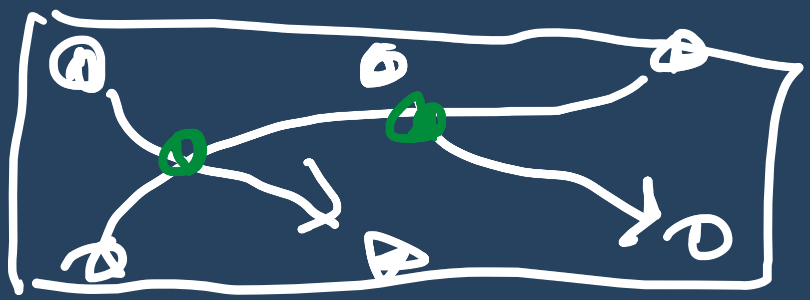
MAGICZNA SZTUCZKA

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

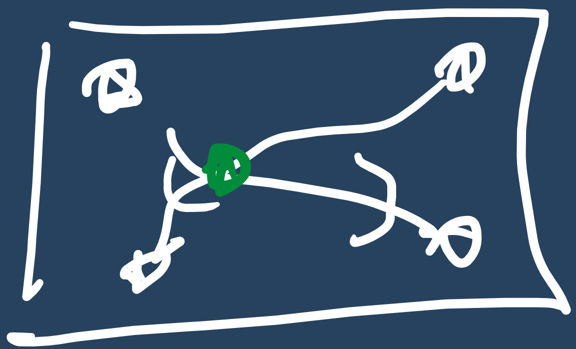



$$\mathrm{sgn}(\pi) = \begin{cases} 1 : & 2 \mid \text{l. przecięć} \\ -1 : & \neg(2 \mid \text{l. przecięć}) \end{cases}$$



sgn = +1

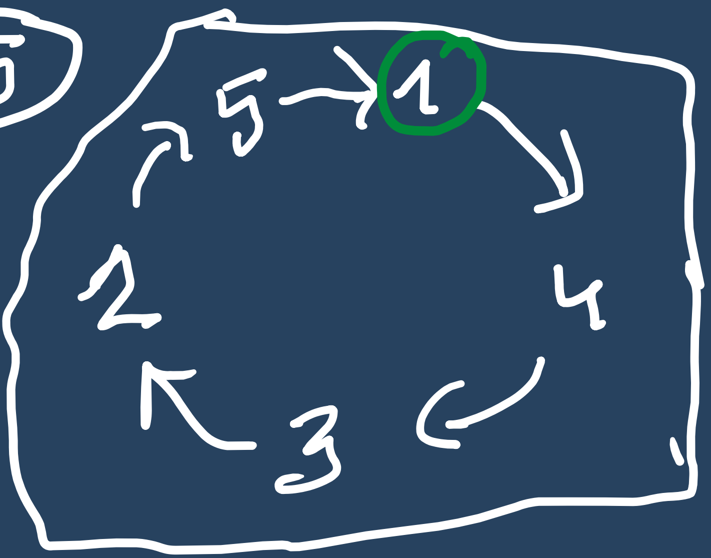



sgn = −1

**FAKT.** Liczba permutacji cyklicznych z $S_n$ jest równa $(n-1)!$

D-d (szkic).

$n = 5$




$$\pi = (1,4,3,2,5) = (4,3,2,5,1) = (3,2,5,1,4)$$

Fix 1 :

$$(n-1)(n-2)\cdot\ldots\cdot 2\cdot 1$$

Obserw: Ustalamy $n$.

- $\mathbb{P}_r(\{\pi \in S_n : \pi \text{ jest cyklem}\}) = \frac{(n-1)!}{n!} = \frac{1}{n}.$

- $n = 100.$

$100! \sim \sqrt{2\pi 100} \cdot \left(\frac{100}{e}\right)^{100}$ ← stirling

$\sim \quad 30^{100}$

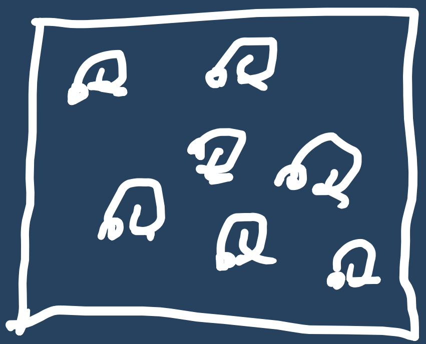
**Def.** Liczby Stirlinga I rodzaju bez znaku. (liczby Stirlinga permutacyjne)

$$\begin{bmatrix} n \\ k \end{bmatrix} = \text{liczba permutacji z } S_n \text{ o } k \text{ cyklach.}$$

- $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$ dla $n \geq 1$
- $\begin{bmatrix} 0 \\ k \end{bmatrix} = [\![ k=0 ]\!]$

- $$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!,$$

- $$\begin{bmatrix} n \\ n \end{bmatrix} = 1$$



- $$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}$$

• $\left[{n \atop 1}\right] = (n-1)!$.

Q : $\left[{n \atop 2}\right] = ?$ $\qquad n \geq 2$


$a+b=n$

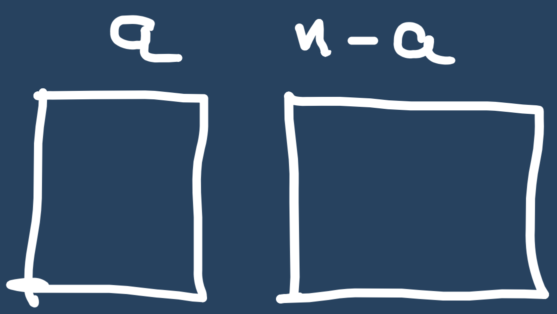
$a \geq 1$

$b \geq 1$

$\binom{n}{a}$

$\left.\begin{matrix} A, A^c \\ A^c, A \end{matrix}\right\}$ musimy uważać aby tego nie zliczać 2 razy

$$\left[{n \atop 2}\right] = \frac{1}{2} \sum_{a=1}^{n-1} \binom{n}{a} (a-1)!\,(n-a-1)!$$




$$= \frac{1}{2} \sum_{a=1}^{n-1} \frac{n!}{a!\,(n-a)!} (a-1)!\,(n-a-1)!$$

$$= \frac{n!}{2} \cdot \sum_{a=1}^{n-1} \frac{1}{a\,(n-a)} = (*)$$

$$\frac{1}{a} + \frac{1}{n-a} = \frac{(n-a)+a}{a(n-a)} = \frac{n}{a(n-a)}$$

$$\frac{1}{a(n-a)} = \frac{1}{n}\left( \; . \; . \; . \; \right)$$

$$(*) = \frac{n!}{2} \sum_{a=1}^{n-1} \frac{1}{n}\left(\frac{1}{a} + \frac{1}{n-a}\right) = \frac{(n-1)!}{2}\left(\sum_{a=1}^{n-1} \frac{1}{a} + \sum \frac{1}{n-a}\right) = \frac{(n-1)!}{2}\left(H_{n-1} + H_{n-1}\right)$$

$$\left[{n \atop 2}\right] = (n-1)!\, H_{n-1} \qquad \blacksquare$$