Liczby partycyjne (II rodzaju) Stirlinga

Wiemy, że

- $$\left\{ {n+1 \atop k+1} \right\} = \left\{ {n \atop k} \right\} + (k+1) \left\{ {n \atop k+1} \right\}$$

- $$x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n+k} \left[{n \atop k} \right] x^{k}$$

$$x^{\overline{n}} = \sum_{k} \left[{n \atop k} \right] x^{k}$$

$$(-x)^{\overline{n}} = (-1)^{n} x^{\underline{n}}$$

FAKT:

$$x^{n} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}$$

Zadanie: udowodnij to

dowód podobny do poprzedniego

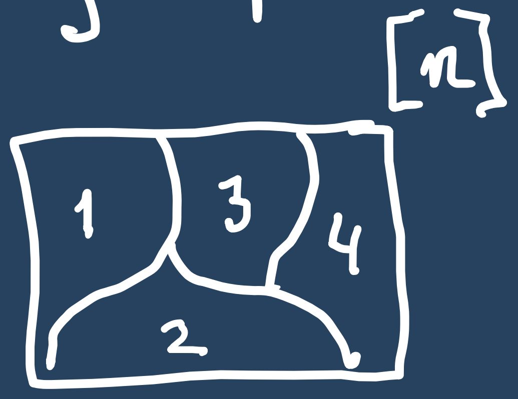
$$x^n = \sum c_{n,k}\, x^{\underline{k}}$$

znajdź rekursję na $c_{n+1,k}$.

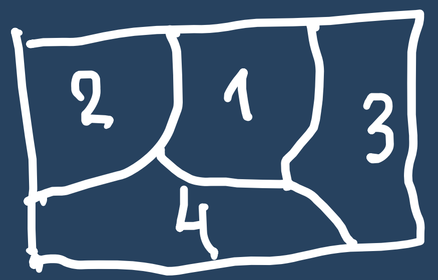
$$\begin{tikzcd}
\{1, x, x^2, x^3, \dots\} \arrow[r, bend left, "{\left[n \atop k \right]}"] & \{1, x^{\underline{1}}, x^{\underline{2}}, x^{\underline{3}}, x^{\underline{4}}, \dots\} \arrow[l, bend left, "{\left\{ n \atop k \right\}}"]
\end{tikzcd}$$

Inny sposób wyznaczania $\left\{ n \atop k \right\}$.

$\mathcal{P}$:



$$f : [n] \xrightarrow{\text{na}} [4]$$



$$f^* : [n] \xrightarrow{\text{na}} [4]$$

$$\mathcal{P} = \{f^{-1}(1), f^{-1}(2), f^{-1}(3), f^{-1}(4)\}$$

$$f \sim f^* \equiv (\exists \pi \in S_4)(f^* = \pi \circ f)$$

WNIOSEK :

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \left| Surj(n, k) \right|$$

zasada włączania-wyłączania

WNIOSEK :

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^i (k-i)^n$$

P

$$\left\{ {n \atop 2} \right\} = \frac{1}{2} \left(2^n - 2 \cdot 1^n\right) = 2^{n-1} - 1.$$

$$\left\{ {n \atop 3} \right\} = \frac{1}{6}\left(\binom{3}{0} 3^n - \binom{3}{1} 2^n + \binom{3}{2} 1^n \right)$$

$$= \frac{1}{6}\left(3^n - 3 \cdot 2^n + 3 \right).$$

LICZBY FIBONACCIEGO

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_{n+2} = F_n + F_{n+1} \end{cases}$$

0, 1, 1, 2, 3, 5, 8, 13, 21 ...

CEL: wyznaczenie zwartego wzoru.

Metoda tradycyjna.

$$\mathcal{F} = \{ x \in \mathbb{R}^{\mathbb{N}} : (\forall n \geq 2)(x_n = x_{n-1} + x_{n-2}) \}$$

- FAKT: $x, y \in \mathcal{F} \longrightarrow$ 1) $x + y \in \mathcal{F}$

  2) $\alpha \cdot x \in \mathcal{F}$.

D-d.

$n \geq 2$: $(x+y)_n = x_n + y_n = x_{n-1} + x_{n-2} + y_{n-1} + y_{n-2} = (x_{n-1} + y_{n-1}) + (x_{n-2} + y_{n-2})$

$= (x+y)_{n-1} + (x+y)_{n-2}$.

- Zgadujemy rozwiązanie:

Szukamy rozwiązania postaci

$$x_n = \lambda^n \quad , \lambda \neq 0$$

$$\lambda^{n+2} = \lambda^{n+1} + \lambda^n \quad / \lambda^n$$

$$\lambda^3 = \lambda + 1$$

$$\lambda^2 - \lambda - 1 = 0 \qquad \Delta = 5$$

$$\lambda_1 = \frac{1 - \sqrt{5}}{2}$$

$$\lambda_2 = \frac{1 + \sqrt{5}}{2}.$$

• szukamy A, B t. żeby

$$y_n = A \cdot \lambda_1^n + B \lambda_2^n$$

była ciąg. Fibb.

Podstawcie $n = 0, n = 1$

$$A + B = 0 \; (= F_0)$$

$$A\lambda_1 + B\lambda_2 = 1 \; (= F_1)$$

ZADANE: dokończ te oblicz.

Metoda funkcji tworzących (generuj.)

$$F(x) = \sum_{n=0}^{\infty} F_n \cdot x^n$$

$$F(x) = 0 \cdot x^0 + 1 \cdot x + \sum_{n \geq 2} F_n x^n =$$

$$= x + \sum_{n \geq 2} (F_{n-1} + F_{n-2}) x^n =$$

$$= x + \sum_{n \geq 2} F_{n-1} x^n + \sum_{n \geq 2} F_{n-2} x^n$$

$$= x + x\sum_{n\geq 2} F_{n-1}x^{n-1} + x^2\sum_{n\geq 2} F_{n-2}x^{n-2}$$

$$= x + x\sum_{k\geq 1} F_k x^k + x^2\sum_{k\geq 0} F_k x^k$$

$$F(x) = x + xF(x) + x^2F(x).$$

$$F(x)(1-x-x^2) = x$$

$$\boxed{F(x) = \frac{x}{1 - x - x^2}}$$

$$x^2 + x - 1 = 0$$

$$\Delta = 5$$

$$\omega_1 = \frac{-1-\sqrt{5}}{2} \qquad \omega_2 = \frac{-1+\sqrt{5}}{2}$$

$$x^2 + x - 1 = (x - \omega_1)(x - \omega_2)$$

- $\omega_1 \cdot \omega_2 = -1$
- $\omega_2 - \omega_1 = \sqrt{5}$

$$x^2 + x - 1 = (x - \omega_1)(x - \omega_2) = (\omega_1 - x)(\omega_2 - x)$$

$$\frac{x}{1 - x - x^2} = -\frac{x}{(\omega_1 - x)(\omega_2 - x)} = \frac{A}{\omega_1 - x} + \frac{B}{\omega_2 - x}$$

$$= \frac{A(\omega_2 - x) + B(\omega_1 - x)}{(\omega_1 - x)(\omega_2 - x)} = \frac{(A\omega_2 + B\omega_1) - (A+B)x}{(\omega_1 - x)(\omega_2 - x)}$$

$$\begin{cases} A + B = 1 \\ A\omega_2 + B\omega_1 = 0 \end{cases}$$

$$\begin{cases} A + B = 1 \\ A\omega_2 + B\omega_1 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ \omega_2 & \omega_1 \end{vmatrix} = \omega_1 - \omega_2 = -\sqrt{5}$$

$$\Delta_A = \begin{vmatrix} 1 & 1 \\ 0 & \omega_1 \end{vmatrix} = \omega_1$$

$$\Delta_B = \begin{vmatrix} 1 & 1 \\ \omega_2 & 0 \end{vmatrix} = -\omega_2 .$$

$$A = \frac{-\omega_1}{\sqrt{5}}$$

$$B = \frac{\omega_2}{\sqrt{5}}$$

$$F(x) = \frac{A}{\omega_1 - x} + \frac{B}{\omega_2 - x} = \frac{A}{\omega_1} \cdot \frac{1}{1 - \frac{x}{\omega_1}} + \frac{B}{\omega_2} \cdot \frac{1}{1 - \frac{x}{\omega_2}}$$

$$= -\frac{1}{\sqrt{5}} \frac{1}{1-\frac{x}{\omega_1}} + \frac{1}{\sqrt{5}} \frac{1}{1-\frac{x}{\omega_2}}$$

$$\sum_{n\geq 0} y^n = \frac{1}{1-y} \qquad |y| < 1$$

$$= -\frac{1}{\sqrt{5}} \sum_{n\geq 0} \left(\frac{x}{\omega_1}\right)^n + \frac{1}{\sqrt{5}} \sum_{n\geq 0} \left(\frac{x}{\omega_2}\right)^n$$

$$= \sum_{n \geq 0} \left(-\frac{1}{\sqrt{5}} \left(\frac{1}{\omega_1}\right)^n + \frac{1}{\sqrt{5}} \left(\frac{1}{\omega_2}\right)^n \right) x^n$$

$$\boxed{= F(x) = \sum_n F_n x^n}$$

$$F_n = \frac{1}{\sqrt{5}} \left(-\left(\frac{1}{\omega_1}\right)^n + \left(\frac{1}{\omega_2}\right)^n \right) =$$

$$= \frac{1}{\sqrt{5}} \left(-\left(-\omega_2\right)^n + \left(-\omega_1\right)^n \right) =$$

$$= \frac{1}{\sqrt{5}} \left(-\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^n \right) =$$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n + (-1)^{n+1}\left(\frac{\sqrt{5}-1}{2}\right)^n\right)$$

wzór Binetta,

$$\begin{cases} F_0 = \frac{1}{\sqrt{5}}(1-1) = 0 \\ F_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} + \frac{\sqrt{5}-1}{2}\right) = 1 \end{cases}$$

UWAGA: $G_0 = 0$; $G_1 = 1$; $G_{n+2} = G_{n+1} + G_n + 1$

metoda twoja: zawodzi:

Jak wyznaczać l. Fibonacciego?

- naiwny pomysł:
  rezerwuję tablicę $F[0,\dots,n]$
  $F[0]=0$
  $F[1]=1$
  for $i=2$ to n do $\{ F[i] = F[i-1] + F[i-2]; \}$
- lepszy pomysł: pamiętaj tylko ostatnie 2 wartości
  $\begin{cases} x=0 \\ y=1 \end{cases}$ w pętli
  $[x,y] = [y, x+y]$

- lepszy (szybszy sposób)

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} F_3 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

Pomysł: zastosuj szybkie potęgowanie do

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

to da nam czas $O(\log n)$

Poprzednie metody: $O(n)$.