Liczby Fibonacciego – II

- $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{M} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$

$$M^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

{ szybkie potęgowanie

- FAKT: $n \geq 1$: $M^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$

  D-d : $n=1$: OK

  Dalej : ind po $n \geq 1$.

WNIOSEK:

$$M^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n\right) = \left(\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)\right)^n$$

$$= (-1)^n$$

$\det(A \cdot B) = \det(A) \cdot \det(B)$

$$\det\left(\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}\right) = F_{n+1} \cdot F_{n-1} - F_n^2$$

$$\boxed{F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n}$$

Tożs. Cassiniego.

Wnioski:

$$M^{n+m} = M^n \cdot M^m$$

$$\begin{bmatrix} F_{n+m+1} & F_{n+m} \\ F_{n+m} & F_{n+m-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \cdot \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix}$$

$$F_{n+m} = F_n \cdot F_{m+1} + F_{n-1} \cdot F_m$$

Obserwacja

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n + (-1)^{n+1}\left(\frac{\sqrt{5}-1}{2}\right)^n\right)$$

$$\frac{1+\sqrt{5}}{2} \sim \frac{3.25}{2} \cong 1.7 > 1$$

$$\frac{\sqrt{5}-1}{2} \sim \frac{1.25}{2} \approx 0.6.. < 1 \qquad \left(\frac{\sqrt{5}-1}{2}\right)^n \xrightarrow[n\to\infty]{} 0$$

$$\boxed{F_n \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n}$$

złota proporcja

$$M^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

$$Mx = \lambda x$$

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

!

wartości własne M:

$$0 = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) =$$

$$= \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda(\lambda-1) - 1 =$$

$$= \lambda^2 - \lambda - 1.$$

$$\lambda^2 - \lambda - 1 = 0 \qquad \Delta = 5$$

$$\omega_1 = \frac{1-\sqrt{5}}{2} \qquad \omega_2 = \frac{1+\sqrt{5}}{2}.$$

Szukamy $x_i \neq 0$ t. że $M\vec{x}_i = \omega_i \cdot \vec{x}_i$

$$f_1 = \begin{bmatrix} \omega_1 \\ 1 \end{bmatrix} \qquad f_2 = \begin{bmatrix} \omega_2 \\ 1 \end{bmatrix}$$

Macierz M w bazie (f_1, f_2)

$$\begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}$$

$$M = P^{-1} \circ \underbrace{\begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}}_{\Delta} \circ P$$

$P : (e_1, e_2) \to (f_1, f_2)$

- $\begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}^n = \begin{bmatrix} \omega_1^n & 0 \\ 0 & \omega_2^n \end{bmatrix}$

- $M^3 = (P^{-1} \circ \Delta \circ \underbrace{P) \circ (P^{-1}}_{Id} \circ \Delta \circ \underbrace{P) \circ (P^{-1}}_{Id} \circ \Delta \circ P)$

$$= P^{-1} \Delta^3 P$$

$$M^n = P^{-1} \cdot \begin{bmatrix} \omega_1^n & 0 \\ 0 & \omega_2^n \end{bmatrix} \cdot P.$$

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

WNIOSEK:

$$F_n = A\omega_1^n + B\omega_2^n$$

- znajdź A, B t.że

$$\begin{cases} A + B = 0 \\ A\omega_1 + B\omega_2 = 1 \end{cases}$$

FUNKCJE TWORZĄCE (Generujące)

- Mamy ciąg $\bar{a} = (a_n)_{n \geq 0}$
- $A(x) = \sum_{n \geq 0} a_n x^n$.

P $a_n \equiv 1$: $F(x) = \sum_{n \geq 0} 1 \cdot x^n = \sum_{n \geq 0} x^n$

$$= \frac{1}{1-x} \quad \text{dla } |x| < 1$$

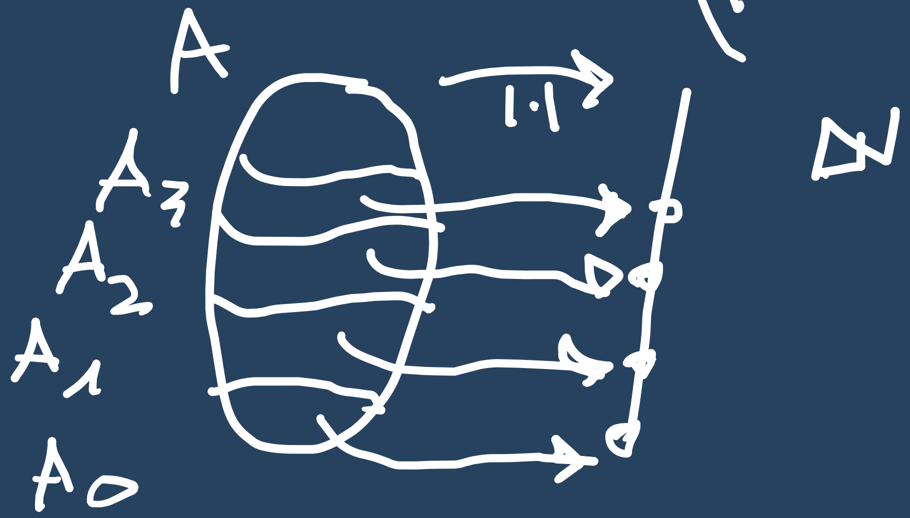
Def. Klasa kombinatoryczna

nazywamy parę $\mathcal{A} = (A, |\cdot|)$ t.że

wagę, koszt

1) $|\cdot| : A \longrightarrow \mathbb{N}$

2) $(\forall n)\left(\left|\{a \in A : |a| = n\}\right| < \infty\right)$.



Oznaczenia:

- $A_n = \{a \in A : |a| = n\}$
- $a_n = |A_n|$

- $\mathcal{A} = (A, |\cdot|) \leftarrow$ kl. komb.

$$\mathcal{A}(x) = \sum_{n \geq 0} a_n \cdot x^n$$

- $[x^n]\, \mathcal{A}(x) = a_n.$

P $\quad \mathcal{N} = (\mathbb{N}, |\cdot|)$, $|n| = n$.

$$\mathcal{N}(x) = \sum_{n \geq 0} |\{k \in \mathbb{N} : |k| = n\}|\, x^n$$

$$= \sum_{n \geq 0} x^n \quad \left(= \frac{1}{1-x} \,,\ |x| < 1 \right)$$

P $\qquad \mathcal{P} = (\{2n : n \in \mathbb{N}\}, |\cdot|) \qquad |x| = x$

$$\mathcal{P}(x) = \sum_{n \geq 0} \#\{k \in \mathcal{P} : |k| = n\} \cdot x^n$$

$$= \sum_{n \geq 0} x^{2n} = \sum_{n \geq 0} (x^2)^n = \frac{1}{1 - x^2}$$

$|x| < 1.$

zadanie: $\mathcal{NP} = (\{2n+1 : n \in \mathbb{N}\}, |\cdot|)$

wyznacz $\mathcal{NP}(x)$.

(P) $\mathcal{A} = \left(\left\{ \underset{1}{\boxed{\cdot}}, \underset{2}{\boxed{\cdot\ \cdot}} \right\}, 1\cdot 1 \right)$ $|\boxed{\cdot}| = 1$

$|\boxed{\cdot\ \cdot}| = 2$

$$\mathcal{A}(x) = 1\cdot x^1 + 1\cdot x^2 = x + x^2.$$

(P) $\tilde{\mathcal{A}} = \left(\left\{ \overset{1}{\boxed{\cdot}}, \underset{1}{\bigodot}, \underset{2}{\boxed{\cdot\ \cdot}} \right\}, 1\cdot 1 \right)$

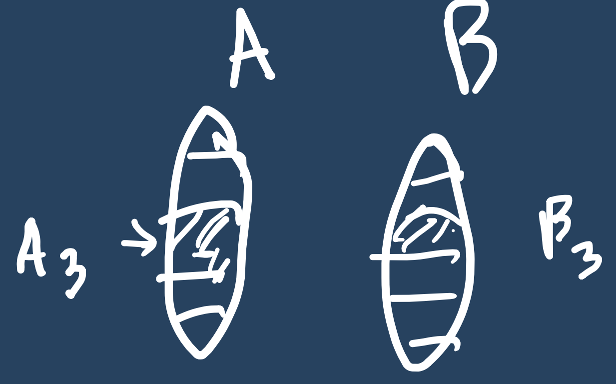
$$\tilde{\mathcal{A}}(x) = 2\cdot x^1 + 1\cdot x^2 = 2x + x^3.$$

Def. Niech $\mathcal{A} = (A, |\cdot|_1)$, $\mathcal{B} = (B, |\cdot|_2)$

oraz $A \cap B = \emptyset$.

Wtedy

$$\mathcal{A} + \mathcal{B} = (A \cup B, |\cdot|),$$



gdzie

$$|x| = \begin{cases} |x|_1 & : x \in A \\ |x|_2 & : x \in B \end{cases}$$

$$((|\cdot| = |\cdot|_1 \cup |\cdot|_2)$$

Obser. Niech $\mathcal{C} = \mathcal{A} + \mathcal{B}$.

Wtedy

$$\sum_n c_n x^n = \sum_n (a_n + b_n) x^n$$

$$= \sum_n a_n x^n + \sum_n b_n x^n$$

$$\mathcal{C}(x) = \mathcal{A}(x) + \mathcal{B}(x)$$

$$(\mathcal{A} + \mathcal{B})(x) = \mathcal{A}(x) + \mathcal{B}(x)$$

Def. $\mathcal{A} = (A, |\cdot|_1)$, $\mathcal{B} = (B, |\cdot|_2)$ ← kl. komb.

Określamy $\mathcal{A} \times \mathcal{B} = (C, |\cdot|)$, gdzie

1) $C = A \times B$

2) $|(a,b)| = |a|_1 + |b|_2$.

$$C_n = \{(a,b) \in A \times B : |(a,b)| = n\} =$$
$$= \{(a,b) \in A \times B : |a|_1 + |b|_2 = n\}$$

$$= \biguplus_{i=0}^{n} \{(a,b) \in A \times B : |a|_1 = i \wedge |b|_2 = n-i\}$$

$$= \biguplus_{i=0}^{n} \left(A_i \times B_{n-i}\right) \qquad \left(= C_n\right)$$

$$c_n = \sum_{i=0}^{n} |A_i \times B_{n-i}| = \sum_{i=0}^{n} a_i \cdot b_{n-i}$$

$$(A \times B)(x) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} a_i \cdot b_{n-i} \right) x^n,$$

$$= A(x) \cdot B(x).$$

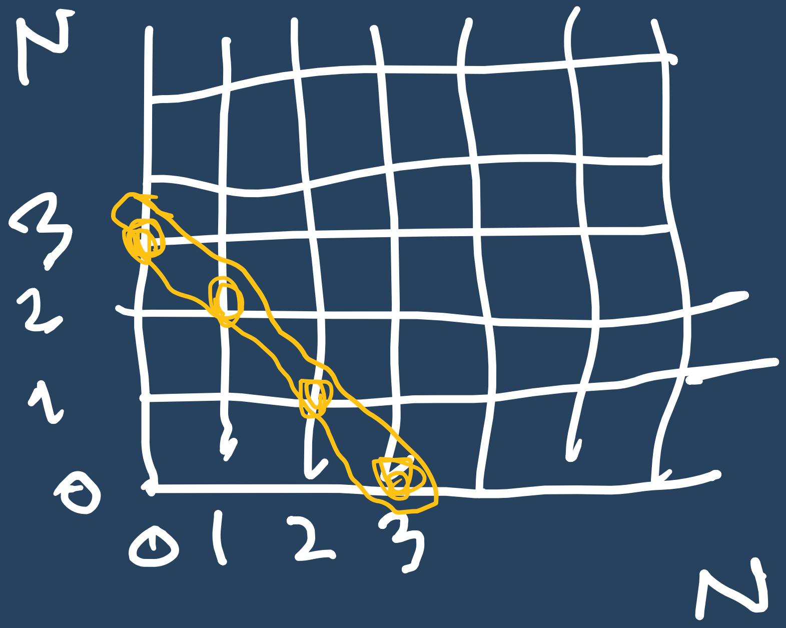
$$\left(\sum_n a_n x^n\right) \cdot \left(\sum_n b_n x^n\right) = \sum_n x^n \sum_{i=0}^{n} (a_i b_{n-i}).$$

$$(A \times B)(x) = A(x) \cdot B(x)$$

P

$$\mathcal{N} = (\mathbb{N}, |\cdot|) \qquad |n| = n$$

$$\mathcal{N}(x) = \frac{1}{1-x}.$$



$$\mathcal{N}\times\mathcal{N} = (\mathbb{N}\times\mathbb{N}, |\cdot|)$$

$$|(a,b)| = a+b.$$

$$(\mathcal{N}\times\mathcal{N})_3$$

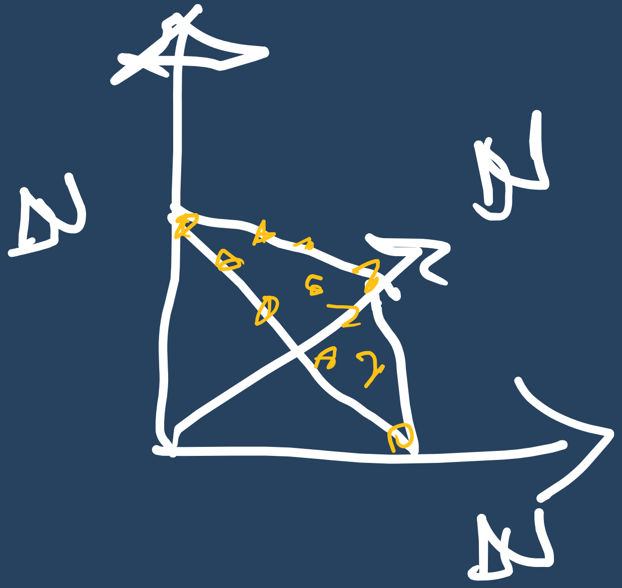
$$(\mathcal{N}\times\mathcal{N})(x) = (\mathcal{N}(x))^2 = \left(\frac{1}{1-x}\right)^2 =$$

$$= \frac{1}{(1-x)^2} = (1-x)^{-2} =$$

$$= \sum_{n\geq 0} \binom{-2}{n} (-x)^n = \sum_{n\geq 0} \binom{n+2-1}{n} (-1)^n \cdot (-1)^n \cdot x^n$$

$$= \sum_{n\geq 0} \binom{n+1}{n} x^n = \sum_{n\geq 0} \binom{n+1}{1} x^n$$

$$= \sum_{n\geq 0} (n+1) x^n \quad ; \quad c_n = (n+1)$$



$$N^3(x) = \frac{1}{(1-x)^3} = \sum_{n \geq 0} \binom{-3}{n} (-1)^n x^n = \sum_{n \geq 0} \binom{n+2}{n} x^n = \sum_{n \geq 0} \underbrace{\frac{(n+2)(n+1)}{2}}_{\binom{n+2}{2}} x^n .$$

uwaga: $\sum_{n\geq 0} n!\, x^n$

promień zbieżn. $= 0$

wykł. funkcje tworzące: $\sum a_n \frac{x^n}{n!}$