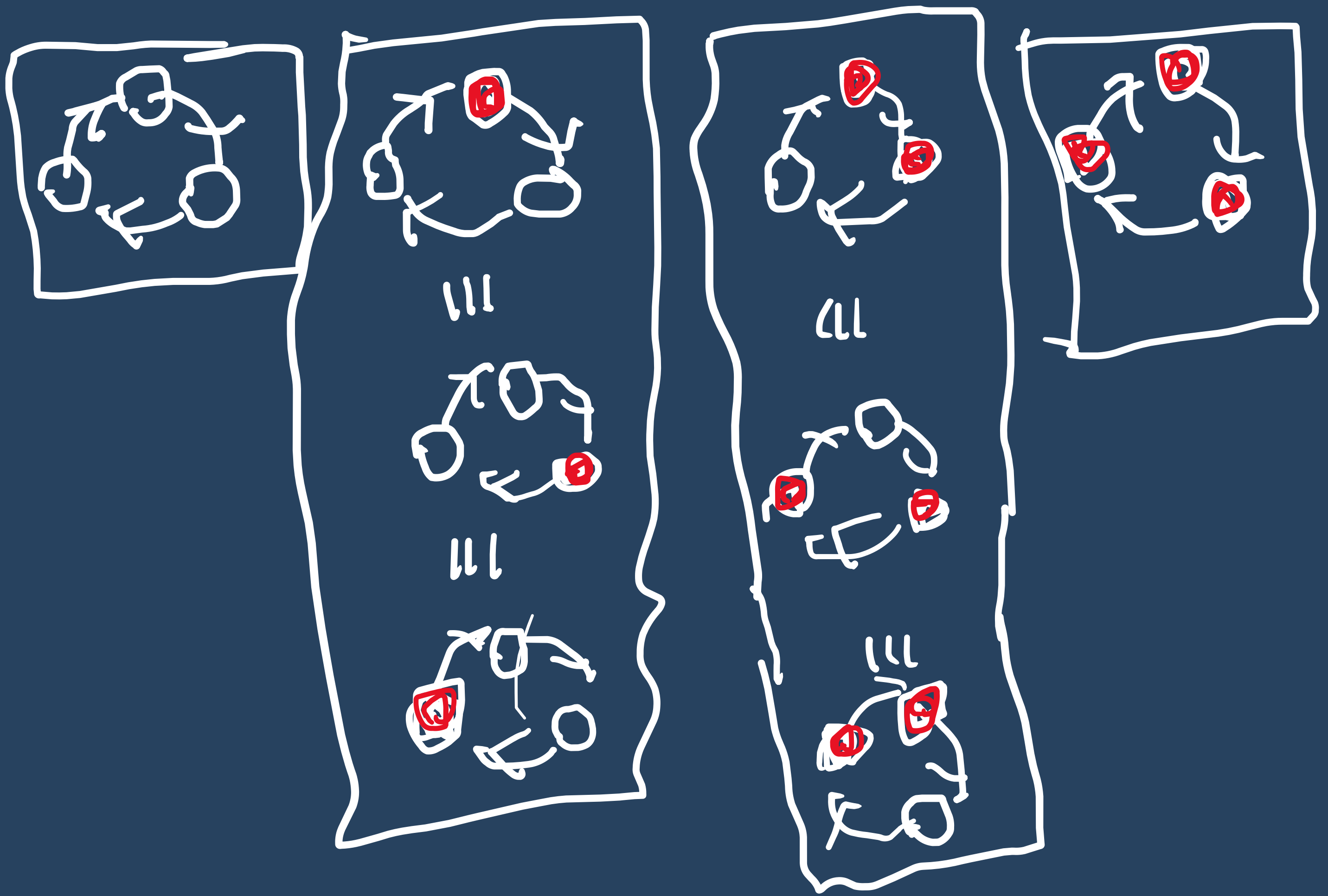
CYKLE

$\mathcal{A} = (A, |\cdot|)$, $a_0 = 0$

Ⓟ $\mathcal{A} = (\{0, \bullet\}, |\cdot|)$ $|\bullet| = |0| = 1$

$n = 3$



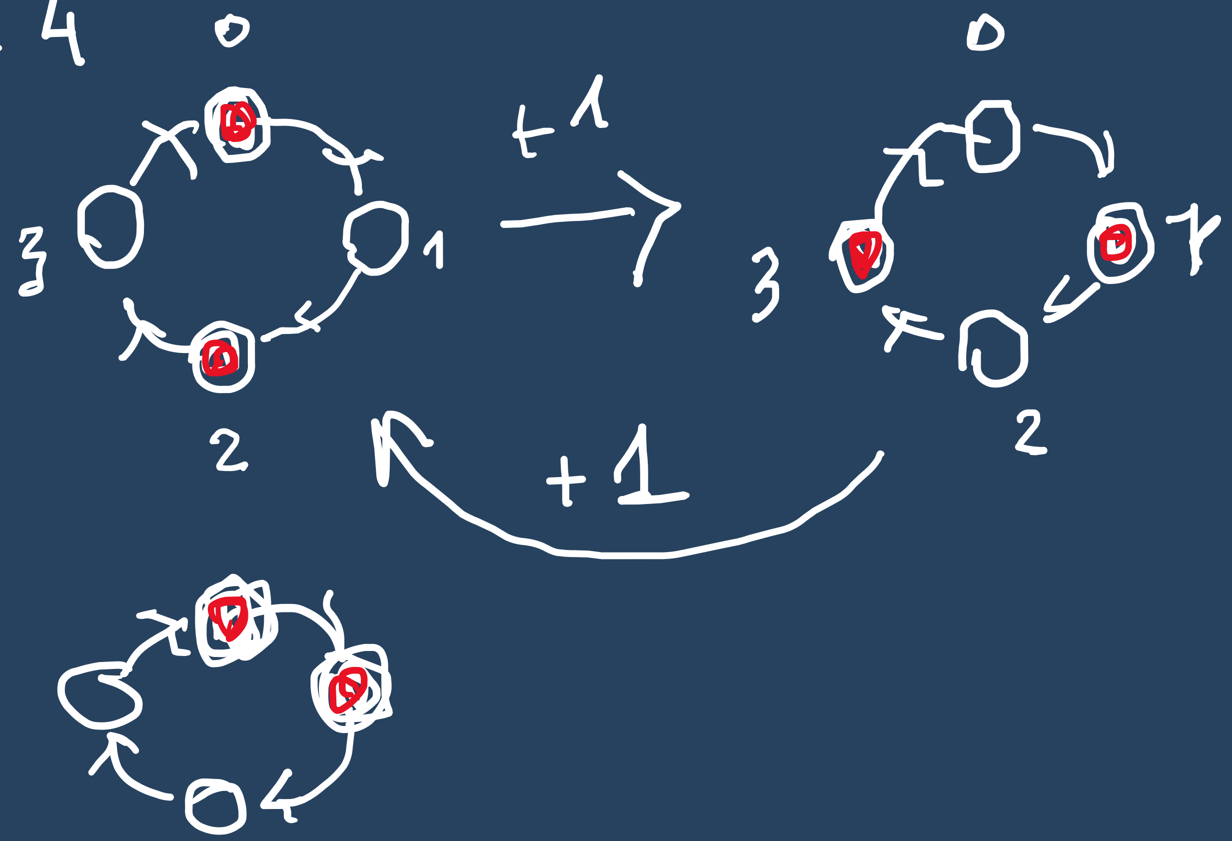
na $\underbrace{A \times \ldots \times A}_{n}$ określamy relację $\sim$

$$(a_0, \ldots, a_{n-1}) \sim_n (b_0, \ldots, b_{n-1})$$
$$\Updownarrow$$
$$(\exists k)(\forall i \in \{0, \ldots, n-1\})\left(b_i = a_{(i+k) \bmod n}\right)$$

zadanie: $\sim_n$ jest rel. równoważ.

$$\mathrm{CYCLE}(A) = \varepsilon + A + (A \times A)/_{\sim_2} + (A \times A \times A)/_{\sim_3} + \ldots$$

(P) $n = 4$



Tu o coś chodzi z podzielnością (?)

Tw. Jeśli $a_0 = 0$, to

$$\mathrm{CYCLE}(\mathcal{A})(x) = \sum_{n \geq 1} \frac{\varphi(n)}{n} \ln \frac{1}{1 - \mathcal{A}(x^n)}$$

$\varphi(n) = |\{k \leq n : \mathrm{NWD}(k,n) = 1\}|$ Funkcja „psi" Eulera.

$\varphi(1) = 1$

$p \in \mathrm{Prime} \rightarrow \varphi(p) = p - 1$.

(P) $\mathcal{A} = (\{0, \circledcirc\}, |\cdot|)$ $\qquad |\circledcirc| = |0| = 1$

$$\mathcal{A}(x) = 2 \cdot x$$

!!! $\ln \frac{1}{1-x} = \sum_{k \geq 1} \frac{1}{k} x^k$

$$(*) = \sum_{n \geq 1} \frac{\varphi(n)}{n} \ln \frac{1}{1 - 2 \cdot x^n} =$$

$$= \sum_{n \geq 1} \frac{\varphi(n)}{n} \sum_{k \geq 1} \frac{1}{k} 2^k x^{n \cdot k} = \sum_{n, k \geq 1} \frac{\varphi(n)}{n \cdot k} 2^k x^{n \cdot k}$$

$$[x^m](*) = \sum_{\substack{n, k \geq 1 \\ n \cdot k = m}} \frac{\varphi(n) \cdot 2^k}{m}.$$

$$[x^3](*) = \sum_{\substack{n \cdot k = 3 \\ n,k \geq 1}} \frac{\varphi(n) \cdot 2^k}{3} =$$

$$= \frac{1}{3} \Big(\underbrace{\varphi(3) \cdot 2^1}_{k=1} + \underbrace{\varphi(1) \cdot 2^3}_{k=3} \Big) =$$

$$= \frac{1}{3}(2 \cdot 2 + 8) = \frac{12}{3} = 4.$$

((ok - zgadza się))

$$[x^5](*) = \frac{1}{5}\left(\underbrace{\varphi(5)\cdot 2^1}_{k=1} + \underbrace{\varphi(1)\cdot 2^5}_{k=5}\right) =$$

$$= \frac{1}{5}(8+32) = 8.$$

$p \in$ Prime: $$[x^p](*) = \frac{1}{p}\left(\varphi(p)\cdot 2 + \varphi(1)\cdot 2^p\right)$$

$$= \frac{1}{p}\left(2p-2+2^p\right).$$

Q: dlaczego to jest l. naturalna?

Zadanie: $\mathcal{B} = (\{\circledcirc\}, |\cdot|)$ $|\circledcirc| = 1$

$B(x) = x.$

CYCLE($\mathcal{B}$): $[x^4]\, \mathrm{Cycle}(B)(x) = 1$

UWAGA. $\mathrm{Seq}(\mathcal{A}) = \mathcal{E} + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + \mathcal{A}^3 + \mathcal{A}^4 + \dots$

$$\mathrm{Seq}(\mathcal{A})(x) = \frac{1}{1 - A(x)}$$

$\mathrm{Seq}_{\geq 1}(\mathcal{A}) = \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + \dots$

$q + q^2 + q^3 + \dots =$

$= q\,(1 + q + \dots)$

$= \dfrac{q}{1-q}$

$$\mathrm{Seq}_{\geq 1}(\mathcal{A})(x) = \frac{A(x)}{1-A(x)}$$

$$\mathrm{Seq}_{\leq n}(\mathcal{A}) = \mathcal{E} + \mathcal{A} + \mathcal{A}^2 + \dots + \mathcal{A}^n$$

$$\mathrm{Seq}_{\leq n}(\mathcal{A})(x) = 1 + A(x) + \dots + A^n(x) =$$

$$= \frac{1 - A^{n+1}(x)}{1-A(x)}.$$

LICZBY FIBONACCIEGO

$$F(x) = \sum_{n \geq 0} F_n x^n$$

$$F(x) = \frac{x}{1 - x - x^2} = x \frac{1}{1 - \underbrace{(x + x^2)}_{q}} = x \frac{1}{1 - q}$$

$$= x \cdot \sum_{n \geq 0} q^n = x \sum_{n \geq 0} (x + x^2)^n = x \sum x^n (1 + x)^n$$

$$= x \sum_{n} x^n \sum_{k} \binom{n}{k} x^k = \sum_{n} \sum_{k} \binom{n}{k} x^{n+k+1}$$

$$[x^n]\, F(x) = F_n$$

$$[x^n]\, F(x) = \sum_{\substack{n,k \\ n+k+1=m}} \binom{n}{k}$$

$n+k=m$

$n=m-k$

$$F_{m+1} = \sum_{n+k=m} \binom{n}{k} = \sum_{k=0}^{m} \binom{m-k}{k}$$

$$F_4 = \sum_{k=0}^{3} \binom{3-k}{k} = \binom{3}{0} + \binom{2}{1} + \underbrace{\binom{1}{2} + \binom{0}{3}}_{0}$$

WSPÓŁCZ. DWUMIANOWE

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

$$F(x,y) = \sum_n (1+x)^n y^n = \sum_n ((1+x)y)^n$$

$$= \frac{1}{1-(1+x)y}$$

szereg Taylora 2 zmiennych

$$F(x,y) = \sum_n \sum_k \binom{n}{k} x^k y^n$$

$$\sum_{n,k} \binom{n}{k} x^k y^n = \frac{1}{1-(1+x)y}$$

Twierdzenie Lagrange'a o inwersji (LIT)

Zał. że $\varphi(u)$ jest funkcją analityczną w $u=0$ oraz, że $\varphi(0)\neq 0$.

rozwij. w szereg potęgowy w 0 o promieniu zbież. >0.

Niech $T(x)=\sum_n t_n x^n$ będzie taka, że

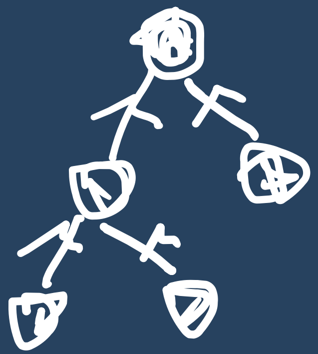
$$T(x)=x\cdot\varphi(T(x))$$

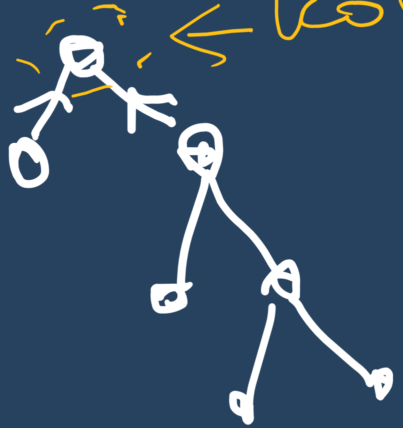
(w otoczeniu 0). Wtedy

$$t_n \left(=[x^n]T(x)\right)=\frac{1}{n}[u^{n-1}](\varphi(u))^n.$$

dla $n\geq 1$.

(P) Drzewa binarne.





korzeń

każdy wierz. ma 0 lub 2

elementy mniejsze

B_n { Ile jest drzew binarnych o n wierzch.?

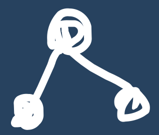
$$B(x) = \sum_n B_n x^n$$

$B_0 = 0$

$B_1 = 1$

$B_2 = 0$

$B_3 = 1$



$\mathcal{B}$ = klasa komb. drzew binarnych

$$\mathcal{B} \cong \{\circ\} + \mathcal{B} \times \{\circ\} \times \mathcal{B}$$

równanie rekurencyjne dla klasy $\mathcal{B}$

$$B(x) = x + B(x) \cdot x \cdot B(x)$$

$$= x \cdot \left(1 + B(x)^2\right)$$

$$= x \cdot \varphi(B(x)),$$

gdzie $\varphi(u) = 1 + u^2$.

$(\varphi(0) = 1 \neq 0)$

$$B_n = \frac{1}{n}[u^{n-1}](1+u^2)^n = \frac{1}{n}[u^{n-1}]\sum_k \binom{n}{k} u^{2k}.$$

- $n-1$ jest niep. $\longrightarrow$ $B_n = 0$

  $\Updownarrow$

  n jest parzyste

- $n-1 = 2m$.

$$[u^{2m}]\sum_k \binom{2m+1}{k} u^{2k}$$

$$B_{2m+1} = \frac{1}{2m+1}\binom{2m+1}{m}$$

uwaga: to zapomniałem dopisać na wykładzie

$$= \frac{1}{2m+1}\,\frac{(2m+1)!}{m!\,(m+1)!} = \frac{(2m)!}{m!\,m!}\cdot\frac{1}{m+1}$$

$$= \binom{2m+1}{m}$$

$$B_{2m+1} = \frac{1}{m+1}\binom{2m}{m} = C_m \quad \blacksquare$$