ON BERNSTEIN SETS

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ABSTRACT. In this note I show a construction of Bernstein subsets of the real line which gives much more information about the structure of the real line than the classical one.

1. BASIC DEFINITIONS

Let us recall that a subset of a topological space is a *perfect set* if is closed set and contains no isolated points. Perfect subsets of real line $\mathbf{R}$ have cardinality continuum. In fact every perfect set contains a copy of a Cantor set. This can be proved rather easilly by constructing a binary tree of decreasing small closed sets. There are continnum many perfect sets. This follows from a more general result: there are continuum many Borel subsets of real.

We denote by $\mathbf{R}$ the real line. We will work in the theory ZFC.

Definition 1. *A subset B of the real line $\mathbf{R}$ is a Bernstein set if for every perfect subset P of $\mathbf{R}$ we have*

$$(P \cap B \neq \emptyset) \wedge (P \setminus B \neq \emptyset) .$$

Bernstein sets are interesting object of study because they are Lebesgue nonmeasurable and they do not have the property of Baire.

The classical construction of a Berstein set goes as follows: we fix an enumeration $(P_\alpha)_{\alpha<\mathfrak{c}}$ of all perfect sets, we define a transfinite sequences (p_α), (q_α) of pairwise different points such that $\{p_\alpha, q_\alpha\} \subseteq P_\alpha$, we put $B = \{p_\alpha : \alpha < \mathfrak{c}\}$ and we show that B is a Bernstein set. This construction can be found in many classical books. The of this note is to give slightly different construction, which will generate simultanously a big family of Bermstein sets.We start with one simple but beautifull result:

Lemma 1. ***(Sierpiński)*** *Suppose that κ is an infinite cardinal number and $\mathcal{A}$ is a family of sets such that $|\mathcal{A}| = \kappa$ and $(\forall A \in \mathcal{A})(|A| = \kappa)$. Then there exists a family $\mathcal{B}$ such that*

(1) $|\mathcal{B}| = \kappa$,
(2) $(\forall B \in \mathcal{B})(|B| = \kappa)$,
(3) $(\forall B, C \in \mathcal{B})(B \neq C \to B \cap C = \emptyset)$,
(4) $(\forall A \in \mathcal{A})(\exists B \in \mathcal{B})(B \subseteq A)$,
(5) $\bigcup \mathcal{B} \subseteq \bigcup \mathcal{A}$

Proof. Let us fix an enumeration $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ and a bijection $\pi = (\pi_1, \pi_2) : \kappa \to \kappa \times \kappa$. Let us fix a well-ordering $\preceq$ of $\bigcup \mathcal{A}$ and let (t_α) be the sequence defined recursively by

$$t_\alpha = \preceq\!-\min(A_{\pi_1(\alpha)} \setminus \{t_\beta : \beta < \alpha\}) .$$

Let us put $B_\alpha = \{t_\beta : \pi_a(\beta) = \alpha\}$ and $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$. This works. □

2. Construction of Bernstein sets

Theorem 1. *There exists a partition $\mathcal{A}$ of the real line $\mathbf{R}$ into continuum sets of cardinality continuum such that each selector of $\mathcal{A}$ is a Bernstein subset of $\mathbf{R}$.*

Proof. Let PERF denotes the family of all perfect subsets of reals. Then $|\mathrm{PERF}| = \mathfrak{c}$ and $(\forall P \in \mathrm{PERF})(|P| = \mathfrak{c})$. Hence we may apply Sirpińskis theorem to the family PERF, so we have a family $\mathcal{C}$ of pairwise disjoint sets of cardinality $\mathfrak{c}$ of sets of cardinality $\mathfrak{c}$ such that

$$(\forall P \in \mathrm{PERF})\exists C \in \mathcal{C})(C \subseteq P) .$$

This is almost this family we would like to get. The only drawback of $\mathcal{C}$ is that its sum may not give the whole real line. So we slightly improve it. Namely, let S be a selector of $\mathcal{C}$ and let

$$\mathcal{A} = \{X \setminus S : X \in \mathcal{C}\} \cup \{S \cup (\mathbf{R} \setminus \bigcup \mathcal{C})\}$$

It is clear that $\mathcal{A}$ is a partition of $\mathbf{R}$ into continuum sets of cardinality continuum. Let $P \in \mathrm{PERF}$ and let S be an arbitrary selector of $\mathcal{A}$. There exists $A \in \mathcal{A}$ such that $A \subseteq P$. But $S \cap A \neq \emptyset$, so also $S \cap P \neq \emptyset$. Let us take anothe selector Q of $\mathcal{A}$ which is disjoint with S. The same argument shows that $Q \cap P \neq \emptyset$. But this implies that $P \setminus S \neq \emptyset$. Therefore S is a Bernstein set. □

3. Conclusions

The following corollary follows immediately from Theorem 1:

Corollary 1. *There exists a partition of the real line into continuum many Bernstein sets*

Let B be a boolean algebra. Let $\mathrm{PD}(B)$ denotes the family of all pairwise disjoint subsets of B, i.e. $X \in \mathrm{PD}(B)$ iff $X \subseteq B \setminus \{0\}$ and $(\forall x, y \in X)(x \neq y \to xy = 0)$. Let finally

$$sat(B) = \min\{\kappa \in \text{card} : (\forall X \in \mathrm{PD}(B))(|X| < \kappa)\}$$

Let $\mathbf{L}$ denote the σ-ideal of Lebesque measure zero subsets of the real line. Let us reall that each Bernstein set is nonmeasurable, hence does not belong to $\mathbf{L}$. From the last Corollary we immediately get the following result:

Corollary 2. $sat(P(\mathbf{R})/\mathbf{L}) \geq \mathfrak{c}^+$

Let

$$\mathrm{non}(\mathbf{L}) = \min\{|X| : X \subseteq \mathbf{R} \wedge X \notin \mathbf{L}\}$$

It is clear that $\omega_1 \leq \mathrm{non}(\mathbf{L}) \leq \mathfrak{c}$. Martin's Axiom implies that $\mathrm{non}(\mathbf{L}) = \mathfrak{c}$ and that $cf(\mathfrak{c}) = \mathfrak{c}$.

Theorem 2. *Suppose that* $cf(\mathfrak{c}) = \mathfrak{c}$ *and* $\mathrm{non}(\mathbf{L}) = \mathfrak{c}$. *Then*

$$sat(P(\mathbf{R})/\mathbf{L}) > \mathfrak{c}^+ .$$

Proof. Let us fix an partition $mathcal A$ from Theorem 1. Let $\mathcal{S}$ be a maximal with respect to inclusion a family of selectors of $\mathcal{A}$ such that

$$X, Y \in \mathcal{S} \wedge X \neq Y \to |X \cap Y| < \mathfrak{c}$$

It is easy to check that $|\mathcal{S}| \geq \mathfrak{c}^+$. Then $\{[S]_{\mathbf{L}} : S \in \mathrm{S}\} \in \mathrm{PD}(P(\mathbf{R})/\mathbf{L})$, hence $sat(P(\mathbf{R})/\mathbf{L}) > \mathfrak{c}^+$. □

It is worth to notice that the assumption $\mathrm{non}(\mathbf{L}) = \mathfrak{c}$ in the last theorem is important. Namely if we extend a model of ZFC+GCH by adding ω_2 random reals then in the resulting model we have $sat(P(\mathbf{R})/\mathbf{L}) = \mathfrak{c}^+$.

4. Final remarks

The construction of a large family of Bernstein sets presented in this note is obviously only a very slight modification of the ingenious orginal one. The transfinite induction from the original one is eliminated and replaced by Sierpiński theorem.