# Recovery of Vertex Orderings in Dynamic Graphs

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***Abstract*—Many networks in the real world are dynamic in nature: nodes enter, exit, and make and break connections with one another as time passes. Several random graph models of these networks are such that nodes have well-defined arrival times. It is natural to ask if, for a given random graph model, we can recover the arrival order of nodes, given information about the structure of the graph. In this work, we give a rigorous formulation of the problem in a statistical learning framework and tie its feasibility, for a broad class of models, to several sets of permutations associated with the symmetries of the random graph model and graphs generated by it. Moreover, we show how the same quantities are fundamental to the study of the information content of graph structures. We then apply our general results to the special cases of the Erdős-Rényi and preferential attachment models to derive strong inapproximability results.**

## I. Introduction

Many networks in the real world (social, biological, technological) are dynamic in nature: that is, nodes enter and leave the network and make and break connections as time passes.

Several random graph models of complex networks incorporate this dynamicity by specifying an evolution process by which the graph grows in time steps. For instance, in each time step of the *preferential attachment* process [1], a new node is added to the network and connections are chosen for it, so that nodes have a well-defined arrival time.

In this work, we investigate how much the *structure* (i.e., the unlabeled version) of a graph generated by such a model tells us about the arrival order of its nodes: in particular, given the graph structure, is it possible to label the nodes with their arrival order, within a certain error tolerance? We formulate this as a statistical learning problem for general random graph models and then provide minimax risk lower bounds in terms of the cardinalities of a few different permutation sets and groups associated with the model and the graphs that it generates. We then apply these general results to a few example random graph models (namely, Erdős- Rényi, which serves as a useful "test bed" for our theory, and preferential attachment, which is somewhat closer to applications). The application to the preferential attachment model leads to nontrivial combinatorial and probabilistic problems, which we partially solve. Finally, we discuss connections to the problem of compression of graph structures.

Though the purpose of this conference paper is primarily to investigate the theoretical aspects of this problem, we speculate that it may find applications in, e.g., the analysis of protein interaction networks, where the order of arrival of nodes in the network may yield information about the robustness of certain biochemical processes. In an epidemiological network (where nodes represent infected agents, and edges indicate contact between two agents), knowing the order of node arrival could be used to prioritize treatment of earlier-arriving nodes. Finally, risk lower bounds for the problem may translate to privacy/anonymity guarantees for individuals in a social network.

We now discuss prior work. While at least one other work has considered the general problem of node arrival order recovery [2], none appear to have done so rigorously and at the same level of generality as we do in the present work. Several related questions have been asked and theoretically studied for preferential attachment graphs and related models; e.g., several authors have considered variations on the problem of determining the first node of a preferential attachment graph [3], [4]. These works are generally restricted to analysis of a particular random graph model, and their setting is qualitatively rather different: namely, they assume that the arrival order of nodes is known, and the goal is to reach the first node via a local exploration process (i.e., the entire graph is not known). In contrast, our results apply (or can be extended) to a broad class of models, and our goal is to recover node arrival order assuming access to the global structure of a graph.

## II. Main results

In Section II-A, we formulate the problem rigorously. First, however, we explain the intuition that leads to this formulation.

One may imagine in an example application that we are presented with a graph on $n$ vertices whose nodes are distinguishable, but not labeled by their arrival order. In order to refer to a given node, we thus assign to each one a unique label in $[n] = \{1, ..., n\}$, resulting in a labeled graph $G'$. Then $G'$ is simply the result of applying a permutation $\pi$ (unknown to us) to the original graph $G$ which is labeled with node

arrival times. Thus, correctly labeling the nodes with their arrival times is precisely equivalent to recovering the inverse of the unknown permutation $\pi$. In particular, the arrival time of the node labeled $j$ in $G'$ is $\pi^{-1}(j)$.

Now, we desire a procedure that does this as well as possible, even when the labeling of $G'$ is chosen in as confusing a way as possible (i.e., adversarially).

From the above remarks, the problem then at least has the following parameters: a random graph model that generates $G$ and an adversary that observes $G$ and generates a permutation $\pi$ to confuse us. In order to evaluate a proposed solution to this problem, we will also need a distortion function that determines the distance between a permutation that our solution guesses and $\pi^{-1}$, the correct answer.

### A. Formulation of the inference problem

To translate the above intuitition to a precise problem formulation, we need a few definitions: for a probability distribution $\mathcal{G}_n$ on graphs on $n$ vertices, a function $A : \mathfrak{G}_n \to S_n$ is called an *adversary function* (here, $\mathfrak{G}_n$ is the set of graphs on $n$ vertices, and $S_n$ is the symmetric group on $n$ letters; graphs in this paper are always labeled, unless explicitly stated otherwise). An *adversary distribution* is simply a probability distribution $\mathfrak{A}_n$ on the set of adversaries (one may think of an adversary distribution as an adversary that can make random choices). A distortion function on the set of permutations is any function $d : S_n \times S_n \to [0, \infty)$. Then we have the following definition:

**Definition 1** (Node age recovery problem)**.** *A* node age recovery problem *is a tuple $(\mathcal{G}_n, \mathfrak{A}_n, d)$; i.e., it is given by a random graph model, a random adversary function, and a distortion measure between two permutations.*

For brevity, we will denote by $\mathfrak{A}_n(G)$, for a graph $G$, a random permutation generated by sampling a random adversary function $A \sim \mathfrak{A}_n$ and applying it to $G$ to produce $A(G)$.

Having defined the parameters of the node age recovery problem, we turn to the form that a solution to it takes: a *node age estimator* is a function $\phi : \mathfrak{G}_n \to S_n$. To evaluate the quality of such an estimator, we need a notion of *risk*. We can define the risk $R$ of an estimator $\phi$ as follows: let $\pi = \mathfrak{A}_n(G)$. Then we define

$$R_{\mathfrak{A}_n}(\phi) = R(\phi) = \mathbb{E}[d(\phi(\pi(G)), \pi^{-1})].$$

For example, if we define $d(\sigma_1, \sigma_2) = d_e(\sigma_1, \sigma_2) = I[\sigma_1 \neq \sigma_2]$, then $R(\phi)$ is simply the probability that $\phi$ fails to recover $\pi^{-1}$ exactly.

We will also be concerned with *approximate recovery*. For this, we define another distortion function: $d_a(\sigma_1, \sigma_2) = \tau(\sigma_1, \sigma_2)$. Here, $\tau(\cdot, \cdot)$ is the *Kendall $\tau$ distance* [5], which is the number of inversions in $\sigma_2 \sigma_1^{-1}$. In other words, it is

$$\sum_{1 \leq i < j \leq n} I[\sigma_2 \sigma_1^{-1}(i) > \sigma_2 \sigma_1^{-1}(j)].$$

In this work, we study the minimax version of the problem: namely, we seek to characterize the minimum, over all choices of estimators, of the maximum risk, over all choices of adversary distributions, for a given random graph model. This is denoted by $R_*(\mathcal{G}_n, d)$. That is,

$$R_*(\mathcal{G}_n, d) = \min_{\phi} \max_{\mathfrak{A}_n} R_{\mathfrak{A}_n}(\phi).$$

*1) A beguiling but flawed formulation:* One might be tempted to formulate the recovery problem in a simpler manner, in which the estimator is only required to recover the original graph $G$. We can illustrate the trouble with such a formulation with an example. Suppose that $n$ people, at distinct time steps, join an initially empty social network and befriend all people already in it. This results in a complete graph on $n$ vertices. Now, a permutation is applied, and we are asked to recover the original graph $G$. Since there is only one graph isomorphic to the complete one on $n$ vertices, we can only give one answer, and it is the correct one. However, despite being correct with probability 1, we have gained no information at all about the order in which the nodes arrived. This is an indication that the problem is fundamentally about recovery of a permutation, rather than of a graph. In fact, one can construct a formulation, which turns out to be exactly equivalent to ours (in the sense that a solution to each formulation can be used to construct a solution to the other, with the same accuracy guarantee, for any choice of distortion measure), in which the explicit goal is to recover a bijection from a set of "names" (i.e., unique identifiers) to the set of possible arrival times (i.e., $\{1, ..., n\}$), with the mapping from the names to nodes in the graph given as a sort of side information.

### B. General bounds on the probability of error

In this section, we give general lower bounds on the probability of error – namely, when we consider the distortion function $d_e$. The bounds are in terms of the sizes of two sets of permutations associated with a random graph model $\mathcal{G}_n$: the first is the *automorphism group* $\mathrm{Aut}(G)$ of a random graph $G$ distributed according to $\mathcal{G}_n$. This is the set of isomorphisms that map $G$ to itself. Since it is defined as a function of $G$, it is a random variable.

The second set, again a function of $G$, we call the set of *feasible permutations* of $G$: it is the subset $\Gamma(G) \subseteq S_n$ which consists of permutations $\sigma$ such that $\sigma(G)$ has positive probability under the distribution $\mathcal{G}_n$.

Related to the feasible permutation set is the set $\mathrm{Adm}(G)$ of *admissible graphs* with respect to $G$: these are simply

the graphs obtained by applying elements of $\Gamma(G)$ to $G$: $\mathrm{Adm}(G) = \{\sigma(G) \ : \ \sigma \in \Gamma(G)\}$.

We have the following relationship, for any graph $G$:

$$|\mathrm{Adm}(G)| = \frac{|\Gamma(G)|}{|\mathrm{Aut}(G)|}. \tag{1}$$

To see this, note first that all elements of $\mathrm{Adm}(G)$ are isomorphic to $G$. Furthermore, the set of isomorphisms from $G$ to $H$ is disjoint from the set from $G$ to $K$, for any $H \neq K \in \mathrm{Adm}(G)$. We thus have

$$|\Gamma(G)| = \sum_{H \in \mathrm{Adm}(G)} |\mathrm{Iso}(G, H)|,$$

where $\mathrm{Iso}(G, H)$ denotes the set of isomorphisms from $G$ to $H$. Finally, it is a general fact that $|\mathrm{Iso}(G, H)| = |\mathrm{Aut}(G)|$ whenever $G$ and $H$ are isomorphic, and equation (1) follows from this.

Having defined the key sets and quantities related to our bounds, we move on to our main results.

In proving a minimax lower bound on the risk, we must design a "hard" adversary. A natural choice is one that simply chooses a permutation uniformly at random from $S_n$ (we will call this the *oblivious* adversary). Against such an adversary, the designer of a node age estimator has two important pieces of information: the graph $\pi(G)$, where $\pi$ is the adversary's permutation, and the fact that $G$ is distributed according to $\mathcal{G}_n$, and is thus a positive-probability graph, and $\mathrm{Adm}(G)$ can be recovered algorithmically given $\pi(G)$. Any lower bound on the risk must thus be constrained by two quantities: $|\mathrm{Aut}(G)|$ and $|\mathrm{Adm}(G)|$. We present such a bound in the next theorem.

**Theorem 1** (General lower bound on the probability of error)**.** *Consider a random graph model $\mathcal{G}_n$ for which any two positive-probability graphs that are isomorphic are equiprobable (this condition may be relaxed; see below), and consider the error probability distortion function $d_e(\sigma_1, \sigma_2) = I[\sigma_1 \neq \sigma_2]$. Then the minimax risk is*

$$\begin{aligned} R_*(\mathcal{G}_n, d_e) &\geq \frac{\mathbb{E}[\log |\mathrm{Aut}(G)|] + \mathbb{E}[\log |\mathrm{Adm}(G)|] - 1}{\log n!} \\ &= \frac{\mathbb{E}[\log |\Gamma(G)|] - 1}{\log n!}. \end{aligned}$$

The condition that any two positive-probability graphs that are isomorphic be equiprobable is satisfied by several random graph models, including, e.g., Erdős-Rényi and preferential attachment. Moreover, this condition can be significantly relaxed, at the expense of a more opaque expression for the lower bound.

### C. Applications to specific random graph models

In this section, we apply our general results to a few different random graph models. In the well known Erdős-Rényi model, denoted by $G(n, p)$, each pair of nodes receives an edge, independently of any other edge event, with probability $p$. Intuitively, since the model itself is entirely symmetric, we expect that the probability of error for any estimator should be quite high, which turns out to be the case.

A less trivial case is the preferential attachment model [1]. There are several definitions, which are slight variations of one another. We will use the following construction of a model $G_m(n)$, for a parameter $m$ an integer $\geq 1$: for $m = 1$, we start at time $t = 1$ with a single vertex with a self-edge. Then, at each time $t > 1$, vertex $t$ joins the graph and chooses a single neighbor from among the vertices $1, ..., t-1$, where vertex $j$ is chosen with probability proportional to its current degree. That is, denoting by $G_{t-1}$ the graph at time $t-1$,

$$\mathbb{P}[t \text{ chooses } j | G_{t-1}] = \frac{\deg_{t-1}(j)}{2t}.$$

This is repeated until there are $n$ vertices in the graph.

We define $G_m(n)$ in terms of $G_1(mn)$ by collapsing each consecutive collection of $m$ vertices into a single one, resulting in a multigraph with loops (we do not discuss the details here, but we need not be overly worried by the presence of multiple edges and loops in our analysis).

In contrast to the situation in the Erdős-Rényi model, in the preferential attachment model there is an intuitive notion of arrival order of vertices. Perhaps surprisingly, exact and approximate recovery (in the sense of Kendall $\tau$ distance) are still almost as difficult as in the Erdős-Rényi case, as the next result formalizes.

**Theorem 2** (Inapproximability for Erdős-Rényi and preferential attachment graphs)**.** *Let $\mathcal{G}_n$ denote either $G(n, p)$ with $p = p(n) \in (0, 1)$ or $G_m(n)$, with $m \geq 3$. Then we have*

$$R_*(\mathcal{G}_n, d_e) = 1 - o(1).$$

*Furthermore, we have, for approximate recovery in the Kendall $\tau$ sense,*

$$R_*(\mathcal{G}_n, d_a) = \Theta(n^2).$$

We remark that the largest possible value of the Kendall $\tau$ distance between two permutations in $S_n$ is $\Theta(n^2)$. Our approximate recovery result implies that, with probability $\Theta(1)$, the Kendall $\tau$ distance between the correct answer and the output of any estimator is as large as possible, up to constant factors.

Furthermore, the error term for both Erdős-Rényi and preferential attachment in the exact recovery case can be bounded explicitly: in the former case, our proof gives $O\left(\frac{1}{n \log n}\right)$ (though a longer, less intuitive proof gives $1/n!$). In the latter, the error term may be bounded above by $O\left(\frac{\log \log n}{\log n}\right)$.

### D. Connection to information content of graph structures

We next discuss a connection to the following problem: fix a labeled random graph model $\mathcal{G}_n$. This induces a distribution on unlabeled graphs (i.e., structures), and we wish to encode samples from this distribution with minimum expected code length. This problem was studied, e.g., for the Erdős-Rényi model in [6]; see also [7], among others. In general, the minimum expected code length is given by $H(S(G))$ (where $G \sim \mathcal{G}_n$ and $S = S(G)$ denotes the unlabeled version of $G$), the entropy of the distribution on structures. Thus, it is of interest to derive asymptotics for $H(S(G))$ and to compare it to $H(G)$. For any random graph model, we have $H(S(G)) = H(G) - H(G|S)$. Now, several random graph models of interest, including Erdős-Rényi and certain formulations of preferential attachment, have the property that all positive-probability labeled representatives of a given unlabeled graph $S$ are equiprobable. In this case,

$$\begin{aligned} H(G|S) &= \mathbb{E}[\log|S|] = \mathbb{E}[\log|\mathrm{Adm}(G)|] \\ &= \mathbb{E}[\log|\Gamma(G)|] - \mathbb{E}[\log|\mathrm{Aut}(G)|], \end{aligned}$$

by the identity (1). Thus, $|\Gamma(G)|$, $|\mathrm{Adm}(G)|$, and $|\mathrm{Aut}(G)|$ are all central quantities in the study of graph structure compression.

## III. Proofs

### A. Proof of Theorem 1

It is sufficient to lower bound the probability of error for a particular adversary. In particular, we consider the one that chooses $\pi$ uniformly at random from $S_n$. To produce the lower bound, we start by applying Fano's inequality: since the goal of an estimator $\phi$ is to estimate $\pi^{-1}$, given $\pi(G)$, we have that the probability of error $p_e = \mathbb{P}[\phi(\pi(G)) \neq \pi^{-1}]$ is

$$p_e \geq \frac{H(\pi^{-1}|\phi(\pi(G))) - 1}{\log|S_n|} = \frac{H(\pi^{-1}|\phi(\pi(G))) - 1}{\log n!}$$

Now, by the data processing inequality, $H(\pi^{-1}|\phi(\pi(G))) \geq H(\pi^{-1}|\pi(G))$. Next, since $\pi^{-1}(\pi(G)) = G$, and $G$ has positive probability under $\mathcal{G}_n$, we must have $\pi^{-1} \in \Gamma(\pi(G))$ (though $\pi(G)$ need not itself have positive probability). Then, since $\pi(G)$ and $G$ are isomorphic, their respective feasible sets $\Gamma(\pi(G))$ and $\Gamma(G)$ are of the same size. Thus, knowing $\pi(G)$, $\pi^{-1}$ is a random variable with distribution having support on a set of size $|\Gamma(G)|$. For a general random graph model, $H(\pi^{-1}|\pi(G))$ may be quite small, depending on the size of $|\mathrm{Aut}(G)|$. By the chain rule, we have

$$\begin{aligned} H(\pi^{-1}|\pi(G)) &= H(G|\pi(G)) + H(\pi^{-1}|G,\pi(G)) \\ &= H(G|\pi(G)) + \mathbb{E}[\log|\mathrm{Aut}(G)|], \end{aligned}$$

since there are exactly $|\mathrm{Aut}(G)|$ isomorphisms between $G$ and $\pi(G)$. Thus, the size of the automorphism group of $G$ plays a role for arbitrary random graph models, and the admissible set of graphs $\mathrm{Adm}(G)$ enters the picture through $H(G|\pi(G))$: namely, if we restrict our attention to models in which all isomorphic graphs with positive probability have the *same* probability, then we have $H(G|\pi(G)) = \mathbb{E}[\log|\mathrm{Adm}(G)|]$, and thus

$$H(\pi^{-1}|\pi(G)) = \mathbb{E}[\log|\mathrm{Adm}(G)|] + \mathbb{E}[\log|\mathrm{Aut}(G)|].$$

Finally, the identity (1) gives us the same formula in terms of $\Gamma(G)$, which completes the proof.

### B. Proof of Theorem 2

For lack of space, we only prove the lower bounds for exact recovery, leaving the proofs of the approximate results to the journal version.

We start with the trivial case of Erdős-Rényi graphs: since any permutation is feasible for any graph in this model, we have that

$$R_*(G(n,p), d_e) \geq \frac{\mathbb{E}[\log n!] - 1}{\log n!} = 1 - \frac{1}{n\log n + O(n)}.$$

That is, with high probability, at least one error will be made by any estimator.

Now we move on to the case of preferential attachment graphs, where we first show a lower bound on the probability of error. In this case, neither quantity appearing in Theorem 1 is easy to compute. However, from [8] and subsequently from [9], we have the following characterization of $|\mathrm{Aut}(G)|$ for fixed $m$: for $m = 1$ (when $G$ is a tree), we have

$$\mathbb{P}[|\mathrm{Aut}(G)| \geq Cn] = 1 - o(1). \tag{2}$$

When $m = 2$, there is at least one symmetry with probability $\Theta(1)$. Finally, when $m \geq 3$, with probability $1 - n^{-\delta}$, for some fixed positive $\delta$, $|\mathrm{Aut}(G)| = 1$ (i.e., $G$ is asymmetric).

Since larger values of $m$ are of more practical interest, we focus on $m \geq 3$. We then have

$$\begin{aligned} \mathbb{E}[\log|\mathrm{Aut}(G)|] &= \mathbb{E}[\log|\mathrm{Aut}(G)|I[|\mathrm{Aut}(G)| > 1]] \\ &\leq n^{-\delta}\log n! = O(n^{1-\delta}\log n) = o(n). \end{aligned}$$

Thus, the automorphisms of $G$ contribute only negligibly to the lower bound on the error probability and to the entropy of the structure of $G$. However, by lower bounding $\mathbb{E}[\log|\Gamma(G)|]$ (hence $\mathbb{E}[\log|\mathrm{Adm}(G)|]$), we will show the claimed error probability lower bound.

*a) A simple lower bound:* Here we show a simple lower bound on $\mathbb{E}[\log|\Gamma(G)|]$, which will imply that the probability of error in the preferential attachment case is $\Theta(1)$. We will then discuss ways of tightening this bound.

Trivially, $\Gamma(G)$ contains all permutations which permute only vertices of degree $m$ in $G$ (we denote the set of vertices

of degree $m$ at time $n$ by $N_n(m)$). With high probability, there are $\Theta(n)$ such vertices. This implies that

$$\mathbb{E}[\log|\Gamma(G)|] \geq \mathbb{E}[I[N_n(m)] \geq Cn] \log|\Gamma(G)|] \\ \geq (1-o(1))\log((Cn)!) = \Theta(n\log n).$$

We thus have that $p_e \geq \Theta(n\log n)/\log n! = \Theta(1)$.

*b) Tightening the lower bound:* We may significantly tighten this bound by a more careful characterization of the structure of the set of feasible permutations and using results from the separate work [9] (in preparation). This is worthwhile in light of the connection to the leading and second-order asymptotics of the structural entropy of preferential attachment graphs, as detailed in Section II-D.

The plan is to consider the directed, acyclic graph version of $G$, denoted by $\mathrm{DAG}(G)$; i.e., its vertex and edge set are the same as in $G$, but each edge in $\mathrm{DAG}(G)$, say, between $u$ and $v > u$, is oriented toward $u$, the older vertex. It turns out that $|\Gamma(G)|$ can be lower bounded in terms of a certain parameter of $\mathrm{DAG}(G)$, as stated in the following lemma.

**Lemma 1** (Characterization of $|\Gamma(G)|$ in the preferential attachment model)**.** *Consider a graph $G$ having positive probability in the preferential attachment model on $n$ vertices. A permutation $\sigma$ is in $\Gamma(G)$ if there is no pair of vertices $v < w$ with a path from $w$ to $v$ in $\mathrm{DAG}(G)$ and $\sigma(v) > \sigma(w)$.*

*Proof.* Suppose that $\sigma \notin \Gamma(G)$. This means that the probability of $H := \sigma(G)$ under the preferential attachment distribution is 0. This can happen only if, for some $t \in [n]$, $\deg_{H,t}(t) \neq m$, where $\deg_{H,t}(j)$ denotes the degree of vertex $j$ at time $t$ in the graph $H$. Consider the smallest such $t$ in $H$. There are exactly $m$ edges coming from $\sigma^{-1}(t)$ in $\mathrm{DAG}(G)$, all of them leading to vertices $< \sigma^{-1}(t)$ (for simplicity of presentation, we will be slightly non-rigorous here and assume that each edge leads to a distinct vertex; however, the correct proof is not too much more complicated). Thus, if $\deg_{H,t}(t) < m$, this means that some neighbor $w < \sigma^{-1}(t)$ in $G$ is mapped by $\sigma$ to a neighbor $\sigma(w) > t$ in $H$.

On the other hand, if $\deg_{H,t}(t) > m$, this means that there must be some neighbor $w > \sigma^{-1}(t)$ in $G$ that is mapped by $\sigma$ to a neighbor $\sigma(w) < t$ in $H$.

In either case, we have a contradiction, which completes the proof. □

In particular, this implies that any product of permutations that only permute vertices within levels (the first level is the set of nodes with in-degree 0, and the $j$th level is the set of nodes with in-edges coming from the $j-1$st level) of $\mathrm{DAG}(G)$ is in $\Gamma(G)$. Thus, we have as a lower bound

$$|\Gamma(G)| \geq \prod_{j\geq 1} |L_j(G)|!, \tag{3}$$

where $L_j(G)$ denotes the collection of vertices in the $j$th level of $\mathrm{DAG}(G)$.

Using this, we can show the following:

**Proposition 1** ( [9] )**.** *We have, for $G \sim \mathcal{G}_m(n)$ with any fixed $m \geq 1$,*

$$\mathbb{E}[\log|\Gamma(G)|] = (1+o(1))n\log n. \tag{4}$$

The full proof is rather detailed (and is given in [9]), so we only sketch it here. The idea is to first show that, with high probability, there are only $o(n)$ vertices occurring in levels $> \log\log n$ (i.e., $\mathrm{DAG}(G)$ is wide, but not deep). Then, among the quite small number of levels containing $n(1+o(1))$ vertices, enough of the $|L_i|$ are large enough (i.e., $\Theta(n)$) for us to apply Stirling's formula to estimate $|L_i|!$, resulting in an estimate of $e^{|L_i|\log|L_i|} = e^{|L_i|\log n + O(n)}$. Multiplying all of these $|L_i|!$, we get

$$|\Gamma(G)| \geq \prod_{j=1}^{\log\log n} |L_i|! = e^{n\log n + O(n)},$$

so that (4) is established. Then the claimed risk lower bound follows from Proposition 1 and Theorem 1.

## IV. Future directions

We have shown inapproximability in terms of Kendall $\tau$ distance for the node age problem for two natural random graph models, but perhaps this distortion measure is overly pessimistic in certain applications; it is worthwhile to consider relaxations of the problem through alternative distortion measures that are useful in certain applied scenarios. As an example, in the preferential attachment model, if we restrict our attention to the very oldest nodes (which are identifiable via their high degrees), the problem becomes much more feasible. One could devise a distortion measure that places emphasis on the proper ordering of the oldest nodes.


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