On Optimal One-dimensional Routing Strategies in Sensor Networks

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Abstract

In this paper we discuss three kind of routing strategies for wireless networks where the positions of nodes are fixed, which is a typical situation in sensor networks. We describe a strategy for finding optimal deterministic strategies and solve explicitly the problem of optimal strategies for some simple but canonical cases when nodes are placed on a line. Next we describe the notion of probabilistic strategies and improve a result from [4] giving a precise formula for a cost of an optimal probabilistic strategies for the nodes placed at equal distances on a line. Finally we introduce the notion of mixed strategies and show some its applications.
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1 Introduction

Wireless sensor networks (WSNs) may become the crucial component of the information infrastructure in industrial control, environmental monitoring and human life rescue operations, as well as in security systems. For this reason a lot of research has been devoted to diverse issues related to deployment of sensor networks. In WSNs, beside the critical research problems such as energy consumptions and network capacity planning, network routing efficiency stands out as the pivotal factor [5].

In this paper we focus on a set of wireless sensors and one *sink* station. The problem that we address here is

> how to route messages from the sensors to the sink so that the energy resources of battery operated sensors are used in the optimal way.

We assume that sensors are not mobile, so we are able to design the optimal routing strategy in advance using high computational power. We do not consider the energy cost of the sink - we suppose that the sink is plugged into an electric network. We do not include in our model the energy costs of sensing and receiving information.

Notation. By $\mathbb{S}$ we denote the set of all sensors (nodes), by $\mathbb{C}$ the set of sinks (called also collectors), and we put $\mathbb{A} = \mathbb{S} \cup \mathbb{C}$. We also assume that the cost of sending a message from x to y is $E(x, y)$, where $E : \mathbb{S} \times \mathbb{A} \to [0, \infty) \cup \{\infty\}$, and $E(x, y) = 0$ if and only if $x = y$. Let $E_{ini} : \mathbb{S} \to (0, \infty)$ be the initial energy of each sensor. Finally, function $I : \mathbb{S} \to (0, \infty)$ describes the number of messages generated by a node in a fixed period T of time (see [1]). Hence a sensor network is described by a tuple $(\mathbb{S}, \mathbb{C}, E, E_{ini}, I)$.

Notice that in the most natural settings the set $\mathbb{A}$ is a subset of three dimensional space $\mathbb{R}^3$. It is usually assumed that (after some rescaling) E is of the form $E(x, y) = (d_e(x, y))^\alpha$ if $d_e(x, y) \le R$ and $E(x, y) = \infty$ if $d_e(x, y) > R$, where d_e is the euclidean distance, α is called exponent, and R is the maximal transmission distance. The value $\alpha = 2$ corresponds to a vacuum and α close to 4 describes a heavily industrialized environment.

We assume that each sensor periodically sends a message to the sink. We assume that the period is the same for all sensors and that initially all sensors have an amount of energy, which is relatively high compared to the cost of sending one message. In this paper we are interested in **maximizing the life-time** of the whole net of sensors - failure of a single node is treated as a failure of the whole network.

The *transmission graph* $\mathrm{Tr}(\mathbb{N})$ of a network $\mathbb{N} = (\mathbb{S}, \mathbb{C}, E, E_{ini}, I)$ is the directed graph $(\mathbb{A}, \tilde{E})$, where $\tilde{E} = \{(x, y) \in \mathbb{S} \times \mathbb{A} : E(x, y) < \infty\}$. We say that $\mathbb{N}$ is *feasible* if for each node $x \in \Omega$ there is a path from x to some sink in $\mathrm{Tr}(\mathbb{N})$.

Let us fix a finite sequence $0 = x_0 < x_1 \le ... \le x_n$, let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and let

$$\mathbf{T}_{\mathbf{x},\alpha,E_{ini}} = (\{x_1, \ldots, x_n\}, \{0\}, |x - y|^\alpha, E_{ini}, 1)\ .$$

We call the structure $\mathbf{T}_{\mathbf{x},\alpha,E_{ini}}$ a **Transmission Line** with n points and one sink with parameter α. In this paper we show some results about this kind of networks. A special kind of such networks is $\mathbb{L}_{n,\alpha,E_{ini}} = \mathbf{T}_{(1,2,\ldots,n),\alpha,E_{ini}}$ and we call this structure a **Simple Transmission Line**. We shall omit in this notation the initial energy E_{ini} if E_{ini} will be constant.

2 Deterministic Strategies

Let $\mathbb{N} = (\mathbb{S}, \mathbb{C}, E, E_{ini}, I)$ be a network and let $\mathrm{Tr}(\mathbb{N})$ be its transmission graph.

Definition 2.1 *A deterministic simple strategy (dss) of the network $\mathbb{N}$ is a function $\sigma : \mathbb{S} \to \mathbb{A}$ such that for each $x \in \mathbb{S}$ the edge $(x, \sigma(x))$ belongs to $\mathrm{Tr}(\mathbb{N})$ and there exists $k \geq 1$ such that $\sigma^k(x) \in \mathbb{C}$.*

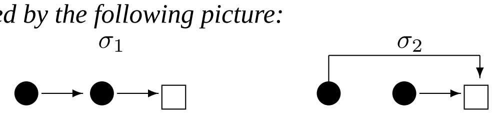
A deterministic strategy is a set of paths in $\mathrm{Tr}(\mathbb{N})$ such that each node corresponds to a single path originating in this node and terminating in some sink.

Simply, a dss σ determines for each node $x \in \mathbb{S}$ the node $\sigma(x)$ where x sends all messages it obtains and the messages generated by itself. The condition $(\forall x \in \mathbb{S})(\exists k \geq 1)(\sigma^k(x) \in \mathbb{C})$ guarantees that each package will be eventually delivered to some sink.

A path of a deterministic strategy determines how to route a message generated in the origin point of the path. Note that now a node may split its outgoing the traffic (in general, the paths need not to merge when they meet). A rationale behind this is that by sending all traffic to a single node one may overload it – the target node has to send out the messages obtained from other nodes and the messages generated by itself.

There is a dss strategy σ for $\mathbf{T}_{(x_1,\ldots,x_n),\alpha}$ that we call **trivial**: $\sigma(x_1) = 0$ and $\sigma(x_k) = x_{k-1}$ for $k = 2, \ldots, n$, i.e. each node sends all messages to the next node towards the sink.

Example 2.1 *Let consider the network $\mathbf{L}_{2,2}$ (the sink placed at point 0, the nodes placed at points $\{1, 2\}$, and $E(x, y) = |x - y|^2$). It is easy to check that there are only two worth to consider strategies, namely dss σ_1 and σ_2 described by the following picture:*



σ_1 is a trivial dss. For strategy σ_1 node 1 uses 2 units of energy and the node 2 uses 1 unit of energy at each round. For strategy σ_2 node 1 uses 1 units of energy and the node 2 uses 4 unit of energy at each round.

2.1 Transformation Into Mixed Integer Linear Programming

Linear Programming. We make use of Linear Programming (LP) technique (see [3]). Mixed Integer Linear Programming (MIP) problem is an LP problem in which some variables are additionally required to be integer. Let us also remark that today there are many powerful software packages that can be used to solve problems LP and MIP, let us mention GLPK, Mathematica, Matlab, LPSolver etc..

One can formulate the problem of finding the optimal deterministic routing strategy as a problem of MIP.

For each node $x \in \mathbb{S}$ and $y \in \mathbb{A}$ we introduce an integer valued variable σ_{xy} denoting the number of messages that are routed from x to y. We introduce also an upper bound D on the energy decay for all nodes for executing the strategy. Then we formulate the following conditions:

- for each sensor the incoming traffic plus the traffic produced by the node is the outcoming traffic:
$$I(x) + \sum_{z \neq x} \sigma_{zx} = \sum_{y \neq x} \sigma_{xy} \ , \tag{1}$$
- the energy decay for each node is at most D, i.e.
$$\sum_{y \neq x} \left(\frac{E(x, y)}{E_{ini}(x)} \cdot \sigma_{xy} \right) \leq D \ . \tag{2}$$

In the above formulation we treat the number $E(x, y)/E_{ini}(x)$ as a constant corresponding to the variable σ_{xy}.

The strategy to derive the optimal routing strategy is to change the values of D and try to solve the MIP problem. A good point is to start with a trivial strategy and then try to reduce D. This is motivated by the fact that for $\alpha \geq 2$ replacing a link to the next node by a longer link increases the energy cost of a sender in a substantial way.

Simple Transmission Line. For $\mathbf{L}_{n,2}$ the trivial dss yields unbalanced energy cost for the sensors: the one located closest to the sink has the cost n to send all messages, the second one has the cost $n - 1$, and so on. The last node has the cost 1. One may try to reduce the maximal cost taking the advantage of the fact that the sensors located at $2, 3, \ldots, n$ have some reserve of energy. So most of these sensors may afford to send a message to a larger distance thereby reducing the cost for the sensors that they are "jumping over".

Nevertheless, we the trivial dss is the optimal simple deterministic strategy.

Theorem 2.1 *For each $n \geq 1$ the trivial routing strategy for the Simple Line $\mathbf{L}_{n,2}$ is an optimal dss, i.e. there is no dss with the maximal cost per node at most $n - 1$.*

Proof. Suppose first that $n > 90$. Let us recall that the cost of the trivial strategy is precisely n. Assume that there exists a strategy with maximal cost per node less than n. Then the node located at 1 cannot transmit all n messages from the network and at least one message comes directly to the sink from some node located at $a \geq 2$. Of course $a < \sqrt{n}$ since otherwise the cost of sending just one message would be at least n.

The messages from nodes from the set $\{a, \ldots, n\}$ must be directed to some nodes from the set $\{0, 1, \ldots, a-1\}$. First notice that less than n/a^2 messages can be transmitted by a. Indeed, the transmission range a is a, i.e. the cost is at least $m \cdot a^2$, where m is the number of messages. If $k \geq 1$ the node $a+k$ can direct less than $n/(k+1)^2$ messages to $\{0, 1, \ldots, a-1\}$. So the total number of messages which may be transmitted by the network is less than

$$a + \frac{n}{a^2} + \sum_{k=1}^{n-a-1} \frac{n}{(k+1)^2} < \sqrt{n} + \frac{n}{4} + n \sum_{k \geq 2} \frac{1}{k^2} =$$

$$\sqrt{n} + n\left(\frac{\pi^2}{6} - \frac{3}{4}\right).$$

But for $n > 90$ we have $\sqrt{n} + n(\frac{\pi^2}{6} - \frac{3}{4}) < n$, hence we obtained a contradiction, which shows that the theorem is true for all $n > 90$.

For all $n \leq 90$ we have run the program glpsol (which is a stand-alone LP/MIP solver) from GLPK package for the MIP problem formulated above and checked that the trivial strategy is optimal. $\square$

Nevertheless, it seems that the trivial strategy is optimal in the class of all deterministic strategy for $\mathbf{L}_{n,2}$ for all $n \geq 1$. Using the program glpsol we check this hypothesis for all $n \leq 200$. Proving this hypothesis seems to be an interesting mathematical problem.

2.2 Optimal Placement of Sensors on a Line

We say that a network satisfies ,,Short Jumps Are Better Hypothesis" (SJBH), if for each $1 \leq a < b \leq n$ we have

$$\frac{|x_a - x_b|^\alpha}{E_{ini}(x_b)} \geq \sum_{k=a}^{b-1} \frac{|x_k - x_{k+1}|^\alpha}{E_{ini}(x_{k+1})} \ . \tag{3}$$

It is easy to check that if $\alpha \geq 1$ and if $E_{ini}(x_k) = 1$ for each k, then the network $\mathbf{T}_{\mathbf{x},\alpha}$ satisfies SJBH.

For a given strategy σ let $\mathrm{cost}(x|\sigma)$ denote the energy decay of sending messages by node x according to strategy σ. Let $\mathrm{cost}(\sigma) = \max_{x \in \mathbb{S}} \mathrm{cost}(x|\sigma)$. Then by **cost** $[\mathcal{S}]$, the cost of $\mathcal{S}$, we mean $\min_\sigma \mathrm{cost}(\sigma)$, where the minimum is taken **over all deterministic strategies** for network $\mathcal{S}$.

Lemma 2.2 *Suppose that $\mathbf{T}_{\mathbf{x},\alpha,E_{ini}}$ satisfies SJBH. Then*

$$\textbf{cost}\left[\mathbf{T}_{\mathbf{x},\alpha,E_{ini}}\right] \geq \frac{1}{n} \sum_{k=1}^{n} (n+1-k) \frac{|x_k - x_{k-1}|^\alpha}{E_{ini}(x_k)} \ .$$

Proof. Let us fix any deterministic strategy η for $\mathbf{T}_{\mathbf{x},\alpha,E_{ini}}$ and let us consider a message sent from a node x_k. After some number of steps this message must reach the sink placed at point 0. Let $a_0 = x_k$, $a_1 = \eta(a_k)$, $\ldots$, $a_m = \eta(a_{m-1}) = 0$. We are going to find a lower bound for $C_k = |a_0 - a_1|^\alpha / E_{ini}(a_0) + \ldots + |a_{m-1} - a_m|^\alpha / E_{ini}(a_{m-1})$. Let $S = \{i < m : a_i > a_{i+1}\}$. Then $C_k \geq \sum\{|a_i - a_{i+1}|^\alpha / E_{ini}(a_i) : i \in S\}$. Moreover $\bigcup\{[a_{i+1}, a_i] : i \in S\}$ covers the whole interval $[0, a_0]$ and the SJBH hypothesis allow us to replace each jump from a_i to a_{i+1} by a sequence of short jumps to immediate predecessors without increasing the value C_k. Hence $C_k \geq \sum_{i=0}^{k-1} |x_{k-i} - x_{k-i-1}|^\alpha / E_{ini}(x_{k-i})$. Hence the sum of energy consumed by all messages is bounded from below by

$$\sum_{k=1}^{n} \sum_{a=1}^{k} \frac{|x_{a-1} - x_a|^\alpha}{E_{ini}(x_a)} = \sum_{k=1}^{n} (n+1-k) \frac{|x_k - x_{k-1}|^\alpha}{E_{ini}(x_a)} \ .$$

Hence at least one node must consume at least $\frac{1}{n}$ fraction of this energy. Therefore

$$\mathrm{cost}(\eta) \geq \frac{1}{n} \sum_{k=1}^{n} (n+1-k) \frac{|x_k - x_{k-1}|^\alpha}{E_{ini}(x_a)} \ ,$$

which finishes the proof. $\square$

2.2.1 Optimal Positions

Let us fix $\alpha \geq 1$, let $E(x-y) = |x-y|^\alpha$. Let us define

$$d_k = \left(\frac{n}{n+1-k}\right)^{\frac{1}{\alpha}} \tag{4}$$

and let

$$x_k = \sum_{i=1}^{k} d_k \ . \tag{5}$$

We consider the network

$$\mathbf{O}_{n,\alpha} = (\{x_1, \ldots, x_n\}, \{0\}, |x-y|^\alpha, 1, 1)$$

where the numbers (x_k) are defined by (5) and the trivial strategy σ. Notice that in particular $x_1 = 1$.

We say that a strategy σ is **balanced** if there exists a constant A such that each $x \in \mathbb{S}$ we have $cost(x|\sigma) = A$.

Theorem 2.3 *For each $n \geq 1$ and $\alpha \geq 1$ the trivial strategy in the network $\mathbf{O}_{n,\alpha}$ is optimal in the class of of all deterministic strategies. Moreover the trivial strategy is balanced and* **cost** $[\mathbf{O}_{n,\alpha}] = n$.

Proof. Let σ be the trivial strategy. Notice that the flow through the kth node in the strategy σ equals $n+1-k$, so

$$\mathrm{cost}(x_k|\sigma) = (n+1-k)(d_k)^\alpha = n \ .$$

Therefore $T(\sigma) = n$ and the strategy σ is balanced. From the assumption $\alpha \geq 1$ one can deduce that the network $\mathbf{O}_n$ satisfies SJBH hypothesis, hence from Lemma 2.2 we get

$$\mathbf{cost}\left[\mathbf{O}_{n,\alpha}\right] \geq \frac{1}{n}\sum_{k=1}^{n}(n+1-k)|x_k - x_{k-1}|^{\alpha} =$$
$$\frac{1}{n}\sum_{k=1}^{n}(n+1-k)\frac{n}{n+1-k} = n\ ,$$

hence $\mathbf{cost}\left[\mathbf{O}_n\right] = n$ and the trivial strategy for $\mathbf{O}_n$ is optimal. $\square$

Theorem 2.4 *Let L_n denotes the position of last node in the network $\mathbf{O}_{n,\alpha}$.*

1. *If $\alpha = 1$, then $L_n = n(\ln n + \gamma) + \frac{1}{2} + O(\frac{1}{n})$,*
2. *If $\alpha > 1$, then $L_n = \frac{\alpha}{\alpha-1}n + \frac{1}{2} + \sqrt[\alpha]{n}\zeta\left(\frac{1}{\alpha}\right) + O(\frac{1}{n})$.*

Proof. Let us recall that $L_n = x_n = \sum_{k=1}^{n} d_k$, so

$$L_n = \sqrt[\alpha]{n}\sum_{k=1}^{n}\frac{1}{\sqrt[\alpha]{n+1-k}} = \sqrt[\alpha]{n}\sum_{k=1}^{n}\frac{1}{\sqrt[\alpha]{k}}\ .$$

The numbers $H_n^r = \sum_{k=1}^{n} k^{-r}$ are called generalized harmonic numbers. We have $H_n^1 = \ln n + \gamma + \frac{1}{2n} + O(\frac{1}{n^2})$, and $H_n^r = n^{-r}(\frac{n}{1-r} + \frac{1}{2} - \frac{r}{12n} + O(\frac{1}{n^3})) + \zeta(r)$ for $0 < r < 1$ (here $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant and $\zeta(z)$ is the Riemann zeta function). Putting these facts together we get the thesis. $\square$

Let us recall that $\zeta(z) < 0$ for $z \in [0,1)$. We also have $\zeta(\frac{1}{2}) \approx -1.46035$. In the case $\alpha = 2$ we get by Thm. 2.4 $L_n \sim 2n + \frac{1}{2} - 1.46035\sqrt{n}$, therefore we are able to build a transmissions line of length almost $2n$ using n sensors in which all sensors use the energy in the same rate when the trivial routing is used.

The network $\mathbf{O}_{n,\alpha}$ is in some sense optimal. Namely, using backward induction one can prove the following result:

Theorem 2.5 *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n$ and let $\alpha \geq 1$. Suppose that $\mathbf{cost}\left[\mathbf{T}_{\mathbf{x},\alpha}\right] \leq n$. Then $x_n \leq L_n$, where numbers L_n are defined as in Thm. 2.4.*

2.2.2 Optimal Distribution of Energy

Finally, let us consider the network

$$\mathbf{S}_{n,\alpha} = (\{1,\ldots,n\}, \{0\}, |x-y|^{\alpha}, E_{ini}, 1)$$

where $E_{ini}(k) = n+1-k$ and the trivial strategy σ. It is clear that this strategy is balanced and $\mathrm{cost}(x|\sigma) = 1$ for each $x \in \{1,\ldots,n\}$.

Theorem 2.6 *If $\alpha \geq 1$ then the trivial strategy for the network $\mathbf{S}_{n,\alpha}$ is an optimal deterministic strategy.*

Proof. It can be shown that the network $\mathbf{S}_{n,\alpha}$ satisfies SJBH hypothesis. Therefore we are able to apply Lemma 2.2 and we get

$$\mathbf{cost}\left[\mathbf{S}_{n,\alpha}\right] \geq \frac{1}{n}\sum_{k=1}^{n}(n+1-k)\frac{1}{E_{ini}(x_k)} = 1\ ,$$

so $\mathbf{cost}\left[\mathbf{S}_{n,\alpha}\right] = \mathrm{cost}(\sigma)$ for the trivial strategy σ. $\square$

3 Probabilistic Strategies

In this section we define a class of routing strategies which is wider that the considered in the previous section deterministic strategies. This strategies are based on probabilistic protocols and implementation of such strategies requires a random number generators of reasonably quality embedded in the environment of each node.

Let $\mathbb{N} = (\mathbb{S}, \mathbb{C}, E, E_{ini}, I)$ be a fixed formal model of sensors network. Let us recall that we put $\mathbb{A} = \mathbb{S} \cup \mathbb{C}$. Let $\mathrm{PD}\left(\mathbb{A}\right)$ denotes that set of all probability distributions on the set $\mathbb{A}$, i.e. $\mathrm{PD}\left(\mathbb{A}\right) = \{p \in [0,1]^{\mathbb{A}} : \sum_{x\in\mathbb{A}} p(x) = 1\}$.

Definition 3.1 *A **probabilistic strategy** for a sensor network $\mathbb{N} = (\mathbb{S}, \mathbb{C}, E, E_{ini}, I)$ is a collection $(p_a)_{a\in\mathbb{S}}$ of elements from $\mathrm{PD}\left(\mathbb{A}\right)$ such that $(\forall a \in \mathbb{S})(p_a(a) = 0)$.*

Notice that each deterministic strategy may be treated as a special case of a probabilistic strategy.

Suppose that the time is divided into discrete time slots. Suppose that at the beginning we have $I(a)$ messages placed at node a. Each message chooses at random a new destination according to the probabilistic strategy $(p_a)_{a\in\mathbb{S}}$: namely if the current position of a message is a node $x \in \mathbb{S}$, then its next position is chosen according to the distribution p_x. If a message falls into some sink, then it stays there forever. We assume also that all choices made by all messages are independent. Let $N(x,y)$ denote the expected number of transmissions from node x to node y, Then the numbers $\mathcal{N} = (N(x,y))_{x\in\mathbb{S},y\in\mathbb{A}}$ have the following properties:
(F1) for all $a \in \mathbb{S}$: $N(a,a) = 0$
(F2) for all $a \in \mathbb{S}$:

$$I(a) + \sum_{x\in\mathbb{S}\setminus\{a\}} N(x,a) = \sum_{y\in\mathbb{A}\setminus\{a\}} N(a,y) \qquad (6)$$

The original probabilities may be recovered from $\mathcal{N}$ by the formula $p_a(y) = N(a,y)/\sum_{z\in\mathbb{A}} N(a,z)$.

The property (F2) can be interpreted as a conservation of flow condition. It can be checked, that when the network $\mathcal{S}$ is feasible, then the above equalities uniquely determine the sequence N. We call a sequence $\mathcal{N} = (N(x,y))_{x\in\mathbb{S},y\in\mathbb{A}}$ a **message flow** on $\mathcal{S}$ if it satisfies (F1) and (F2).

Similarly as in the case of dss we define the *cost function* of a message flow $\mathcal{N}$ by the equation $\mathrm{cost}(a|\mathcal{N}) = \sum_{y\in\mathbb{A}} \frac{E(a,y)}{E_{ini}(a)} N(a,y)$. Our goal is to minimize the function

$$\max_{a\in\mathbb{S}} \mathrm{cost}(a|\mathcal{N}) \tag{7}$$

over the class of all messages flows $\mathcal{N}$ over network $\mathcal{S}$.

3.1 Translation into LP Problem

We follow [1], [2] and translate this problem into the language of Linear Programming. For this goal we introduce one variable D for upper bound for energy decay and we use variables $(N_{xy})_{x\in\mathbb{S},y\in\mathbb{A}}$ corresponding to values $(N(x,y))_{x\in\mathbb{S},y\in\mathbb{A}}$. Our goal (7) translates into the following linear programming problem:

$$\begin{cases} \text{goal} & \text{minimize D} \\ \text{s.t.} & (\forall a \in \mathbb{S}) \left(\sum_x N_{xa} + I(a) = \sum_y N_{ay}\right) \\ & (\forall a \in \mathbb{S}) \left(D \geq \sum_y \frac{E(a,y)}{E_{ini}(a)} N_{ay}\right) \\ & (\forall a \in \mathbb{S})(\forall y \in \mathbb{A})(N_{ay} \geq 0) \end{cases} \tag{8}$$

Let us also remark that we are able to model this situation when some sensors from considered network have an upper bound on transmission range. We can use two strategies: (1) we can eliminate variables N_{xy} from our model when y is not accessible from x, (2) we can add an additional equation $N_{xy} = 0$ when y is not accessible from x.

If the network $\mathcal{S}$ is feasible, then there is a subforest of $\mathrm{Tr}(\mathcal{S})$ with roots in $\mathbb{C}$. This forest defines a dss from which we obtain one feasible message flow satisfying (8). So there are feasible solutions of the problem (8).

We say that a message flow $\mathcal{N}$ is **balanced**, if there exists a constant c such that $(\forall a \in \Omega)(\mathrm{cost}(a|\mathcal{N}) = c)$. Balanced flows use evenly energetic resources of all sensors from the network, hence optimal balanced solutions are the preferred ones in planning of routing strategies.

Example 3.1 *Let us consider the network $\mathbf{T}_{(0.1,0.2,0.9,1),2}$ The LP solver glpk gives the following probabilistic strategy*

$$(p_{a,b})_{\substack{a=1..4\\b=0...4}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.866071 & 0.133929 & 0 \end{pmatrix}$$

Hence there are the following jumps: $0.01 \to 0$, $0.2 \to 0$, $0.9 \to 0.2$, *and* $1 \to 0.9$ *(with probability 0.133929),* $1 \to 0.2$ *(with probability 0.866071). The costs of the nodes are, respectively, 0.01, 0.12, 0.555625, 0.555625, hence the cost of this network is 0.555625.*

The cost of a trivial strategy is 0.98 in this case. The cost of an optimal deterministic strategy defined by $\{1 \to 0.2, 0.8 \to 0.2, 0.2 \to 0, 0.1 \to 0\}$ *is 0.64.*

3.2 Simple Transmission Line

We consider once again the Simple Transmission Line $\mathbf{L}_{n,\alpha}$ The following theorem was proved in [4]:

Theorem 3.1 *For each $n > 0$ there is an optimal balanced probabilistic strategy for the structure $\mathbb{L}_{n,2}$. Moreover, there are only two kind of hops in the optimal solution: to the sink and to the previous node.*

We shall explicitly describe the optimal strategy for the structure $\mathbb{L}_n$ and prove the following result:

Theorem 3.2 *The cost in the optimal strategy for $\mathbb{L}_{n,2}$ is*

$$n - ((1 + \frac{1}{n})H_n - 2)\ , \tag{9}$$

where H_n denotes the nth Harmonic Number.

Let us notice that $(1 + \frac{1}{n})H_n - 2 > 0$ for each $n > 1$, hence the cost of the optimal strategy for the structure $\mathbb{L}_{n,2}$ is smaller than n.

Proof. From Thm. 3.1 we know that the optimal strategy for $\mathbb{L}_n$ is balanced and we know its architecture. Let N_i denote the flow from the ith node to the $(i-1)$th node and let M_i denote the flow from the ith node to the sink 0. Notice that $M_1 = 0$. Moreover for each $i = 1, \ldots n-1$ we have

$$N_i + M_i = 1 + N_{i+1} \tag{10}$$

and $N_n + M_n = 1$. If we put additionally $N_{n+1} = 0$, then the equation $N_i + M_i = 1 + N_{i+1}$ holds for all $i = 1, \ldots, n$.

Notice that the cost of the ith node equals $N_i + i^2 M_i$. Since the optimal flow is balanced, we get

$$N_i + i^2 M_i = N_{i+1} + (i+1)^2 M_{i+1} \tag{11}$$

From Eq. 10 and 11 we deduce that $(i+1)^2 M_{i+1} = 1 + (i^2 - 1)M_i$ and this recurrence can be solved explicitly, giving us $M_k = (k - H_k)/((k-1)k)$. From this we deduce that the cost the last node equals

$$N_n + n^2 M_n = 1 - M_n + n^2 M_n = \\ 2 + n - \frac{(1+n)H_n}{n}\ . \qquad \square$$

A standard approximation of the number H_n yields the following result:

Corollary 3.1 **cost** $[\mathbb{L}_{n,2}] = n - \ln n + (2 - \gamma) + O(\frac{\ln n}{n})$

Notice that the cost of one round of transmission in the trivial routing on the structure $\mathbb{L}_{n,2}$ is precisely n. Therefore, the use of the optimal probabilistic routing strategy described in [4] give us an improvement of order $\ln n$. Hence the sense of using this kind of strategies **is problematic** in this case: we get a slightly better strategy but the protocol is more complicated. In fact, using the optimal probabilistic strategy we obtain only one property - all nodes in the network die more of less precisely in the same time, while in the trivial strategy the first node (i.e. the node with number 1) dies first.

Let us consider the optimal probabilistic routing on the structure $\mathbb{L}_{n,2}$. Let $p_{n,k}$ denote the probability of routing from the kth node directly to the sink placed at point 0. Using the above notation we see that $p_{n,k} = \frac{M_k}{N_k+M_k}$. Since $M_k = (k-H_k)/((k-1)k)$ and $N_k + k^2 M_k = 2+n-(1+1/n)H_n$ we get

$$p_{n,k} = \frac{k-H_k}{(k-1)k((n+1-k)+(1+\frac{1}{k})H_k-(1+\frac{1}{n})H_n)} \ .$$

From this equation we can, for example, deduce that if n tends to infinity then $p_{n,n} \sim \frac{1}{n}$. This observation shows some threats of using probabilistic strategies. Namely, if the initial energy E_{ini} is relatively small (for example if $E_{ini}(x) = n^2$, i.e. there is a sufficient account of energy for n rounds of collecting messages by a sink), then due to probabilistic uncertainty with a quite high probability the last node will exhaust its energy before the nth round. The problem in this example disappears, if the initial energy is essentially larger, for example if $E_{ini}(x) = n^3$ and $n \geq 10$.

4 Mixed Strategies

A mixed strategy is a strategy which is a probabilistic combination of deterministic simple strategies. Instead of giving precise mathematical definition we shall illustrate this concept by two examples.

Example 4.1 *In Example 2.1 we considered two strategies σ_1 and σ_2 for the network $\mathbf{L}_{2,2}$. We checked that $\mathrm{cost}(\sigma_1) = 2$ and $\mathrm{cost}(\sigma_2) = 4$. Let us consider the strategy $\eta = \frac{3}{4}\sigma_1 + \frac{1}{4}\sigma_2$. A simple implementation of this strategy is a „round-robin" way: we use 3 times σ_1, then 1 time σ_2 and so on. The cost of this mixed strategy is $1\frac{3}{4}$. In fact η is an optimal probabilistic strategy for $\mathbf{L}_{2,2}$.*

Example 4.2 *In Example 3.1 we found an optimal probabilistic strategy for the structure $\mathbf{T}_{(0.1,0.2,0.9,1),2}$. Let us consider the following two dss for this network: $\sigma_1 = \{1 \to 0.2, 0.8 \to 0.2, 0.2 \to 0, 0.1 \to 0\}$ and $\sigma_2 = \{1 \to 0.8, 0.8 \to 0.2, 0.2 \to 0, 0.1 \to 0\}$. The σ_1 is an optimal dss and $\mathrm{cost}(\sigma_1) = 0.64$. It is easy to check that and $\mathrm{cost}(\sigma_2) = 0.98$. Let $\eta = \frac{8}{9}\sigma_1 + \frac{1}{9}\sigma_2$. The cost of this mixed strategy is 0.577 and the cost of the optimal probabilistic strategy was 0.555625. As before, this strategy can be implemented in a „round-robin" way: we use 8 times σ_1, then 1 time σ_2 and so on.*

The above example shows that the notion of mixed routing strategy with fractional probabilities with small denominators are easy for implementations and may be useful as an approximations of optimal probabilistic strategies.

Further Works

It would be desirable to extend our investigations to two dimensional structures. However, this kind of networks produces large LP problems consuming large computational resources. Therefore it is essential to reduce size of these problems by using, for example, symmetries of considered networks or any theoretical information about the shape of optimal solutions.

Another interesting area is the problem of approximation of an arbitrary probabilistic routing strategy by mixed strategies, which as we shown in Section 4 are easier to implement and behave in a more predictable way than the pure probabilistic strategies.

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References

[1] J. H. Chang and L. Tassiulas. Routing for maximum system lifetime in wireless ad-hoc networks. In *Proc. of 37th Annual Allerton Conference on Communication, Control, and Computing*, September 1999.

[2] J. H. Chang and L. Tassiulas. Energy conserving routing in wireless ad-hoc networks. In *Proc. IEEE INFOCOM*, page 2231, March 2000.

[3] G. Dantzig. *Linear programming and extensions*. Princeton, NJ, Princeton Univ. Press, 1963.

[4] A. Giridhar and P. R. Kumar. Maximizing the functional lifetime of sensor networks. In *IPSN '05: Proceedings of the 4th international symposium on Information processing in sensor networks*, Piscataway, NJ, USA, 2005. IEEE Press.

[5] A. Jarry, P. Leone, O. Powell, and J. a. Rolim. An optimal data propagation algorithm for maximizing the lifespan of sensor networks. *Distributed Computing in Sensor Systems*, pages 405–421, 2006.