The Reconstruction of Polyominoes from Approximately Orthogonal Projections*

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Abstract. The reconstruction of discrete two-dimensional pictures from their projection is one of the central problems in the areas of medical diagnostics, computer-aided tomography, pattern recognition, image processing, and data compression. In this note, we determine the computational complexity of the problem of reconstruction of polyominoes from their approximately orthogonal projections. We will prove that it is NP-complete if we reconstruct polyominoes, horizontal convex polyominoes and vertical convex polyominoes. Moreover we will give the polynomial algorithm for the reconstruction of hv-convex polyominoes that has time complexity $O(m^3n^3)$.

1 Introduction

1.1 Definition of Problem

A finite binary picture is an $m \times n$ matrix of 0's and 1's, when the 1's correspond to black pixels and the 0's correspond to white pixels. The i-th *row projection* and the j-th *column projection* are the numbers of 1's in the i-th row and of 1's in the j-th column, respectively. In a reconstruction problem, we are given two vectors $H = (h_1, \ldots, h_m) \in \{1, \ldots, n\}^m$ and $V = (v_1, \ldots, v_n) \in \{1, \ldots, m\}^n$, and we want to decide whether there exists a picture which the i-th row projection equals h_i and which j-th column projection equals v_j.

Often, we consider several additional properties like symmetry, connectivity or convexity. In this paper, we consider three properties:

horizontal convex (*h-convex*) — in every row the 1's form an interval,

vertical convex (*v-convex*) — in every column the 1's form an interval, and

connected — the set of 1's is connected with respect to the adjacency relation, where every pixel is adjacent to its two vertical neighbours and to its two horizontal neighbours.

* Supported by KBN grant 8 T11C 043 19

A connected pattern is called a *polyomino*. A pattern is *hv-convex* if it is both h-convex and v-convex.

In this paper we solve the problem (Woeginger [5]) whether there exists a polynomial time algorithm that takes as an input a horizontal projection vector $H \in \mathbb{R}_+^m$ and a vertical projection vector $V \in \mathbb{R}_+^n$, and which outputs a polyomino whose projections $H^* \in \{1, \dots, n\}^m$ and $V^* \in \{1, \dots, n\}^m$ approximate the vectors H and V, respectively. We consider two notions of "approximate"

(1) every component of (H, V) differs by at most one from the corresponding component of (H^*, V^*), i.e. we select only the nearest positive integers (we call this version *the approximation with the absolute error*), and

(2) for every h_i and v_j it is $|h_i - h_i^*| \leq \log(h_i + 1)$ and $|v_j - v_j^*| \leq \log(v_j + 1)$ for $i = 1, \dots, m$ and $j = 1, \dots, n$ (*the approximation with the logarithmic error*).

The algorithm outputs "NO" if there does not exist a polyomino with approximate projections V and H.

1.2 Known Results

First Ryser [6], and subsequently Chang [2] and Wang [7] studied the existence of a pattern satisfying orthogonal projections (H, V) in the class of sets without any conditions. They showed that the decision problem can be solved in time $O(mn)$. These authors also developed some algorithms that reconstruct the pattern from (H, V).

Woeginger [8] proved that the reconstruction problem in the class of polyominoes is an NP-complete problem. Barcucci, Del Lungo, Nivat, Pinzani [1] showed that the reconstruction problem is also NP-complete in the class of h-convex polyominoes and in the class v-convex polyominoes.

The first algorithm that establishes the existence of an hv-convex polyomino satisfying a pair of assigned vectors (H, V) in polynomial time was described by Barcucci et al. in [1]. Its time complexity is $O(m^4n^4)$ and it is rather slow. Gębala [4] showed the faster version of this algorithm with complexity $O(\min(m^2, n^2) \cdot mn \log mn)$. The latest algorithm described by Chrobak and Dürr in [3] reconstructs the hv-convex polyomino from orthogonal projection in time $O(\min(m^2, n^2) \cdot mn)$.

All above results concern to the reconstruction polyominoes from exact orthogonal projections (RPfOP).

1.3 Our Results

In this paper we study complexity of the problem of reconstruction polyominoes from approximately orthogonal projections (RPfAOP). In section 2 we prove that RPfAOP (for both kinds of errors) is NP-complete in the classes of (1) polyominoes, (2) horizontal convex polyominoes and (3) vertical convex polyominoes. In section 3 we show that RPfAOP, for an arbitrary chosen function of error, is in P for the class of hv-convex polyominoes. We describe an algorithm that solves this problem and has complexity $O(m^3n^3)$.

2 Hardness

In this section we show the reduction from the problem of reconstruction of polyominoes from exact orthogonal projections (RPfOP) to the problem of reconstruction of polyominoes from approximately orthogonal projections. Let

$$\tilde{H} = \{\tilde{h}_1, \ldots, \tilde{h}_m\} \in \{1, \ldots, n\}^m,$$

$$\tilde{V} = \{\tilde{v}_1, \ldots, \tilde{v}_n\} \in \{1, \ldots, m\}^n$$

be an instance of RPfOP problem. Moreover we assume that

$$\sum_{i=1}^{m} \tilde{h}_i = \sum_{j=1}^{n} \tilde{v}_j,$$

otherwise the polyomino with the projections $(\tilde{H}, \tilde{V})$ does not exist. By this instance, we will construct a row vector $H = \{h_1, \ldots, h_m\} \in \mathbb{R}_+^m$ and a column vector $V = \{v_1, \ldots, v_n\} \in \mathbb{R}_+^n$ adequate to the notion of error. For the approximation with the absolute error we fix

$$h_i = \tilde{h}_i - \frac{1}{2}, \qquad i = 1, \ldots, m,$$

$$v_j = \tilde{v}_j + \frac{1}{2}, \qquad j = 1, \ldots, n.$$

And for the logarithmic approximation we choose h_i such that

$$\tilde{h}_i \le h_i + \log(h_i + 1) < \tilde{h}_i + 1,$$

and v_j such that

$$\tilde{v}_j - 1 < v_j - \log(v_j + 1) \le \tilde{v}_j.$$

The choice is always possible because functions $x - \log(x+1)$ and $x + \log(x+1)$ are continuous and strictly increasing surjections on $\mathbb{R}_+$.

Now we can solve the RPfAOP problem for projections (H, V).

Lemma 1. *If there exists a polyomino P with row projections $H^* \in \{1, \ldots, n\}^m$ and column projections $V^* \in \{1, \ldots, m\}^n$, such that (H^*, V^*) is the approximation with the absolute (logarithmic) error of (H, V), then there exists a polyomino with projections $(\tilde{H}, \tilde{V})$.*

Proof. For the polyomino P the following properties hold

(i) $\sum_i h_i^* = \sum_j v_j^*$ (the sums are equal to number of 1's in polyomino P), and

(ii) $|h_i^* - h_i| \le 1$ and $|v_j^* - v_j| \le 1$ for the absolute error ($|h_i^* - h_i| \le \log(h_i + 1)$ and $|v_j^* - v_j| \le \log(v_j + 1)$ for the logarithmic error).

But from the definition of (H, V) the above properties occur if and only if for all i we have that h_i^* is equal to the maximal admissible value, i.e.

$$h_i^* = \lfloor h_i + 1 \rfloor = \tilde{h}_i \qquad (\; h_i^* = \lfloor h_i + \log(h_i + 1) \rfloor = \tilde{h}_i \;),$$

and for all j we have that v_j^* is equal to the minimal admissible value, i.e.

$$v_j^* = \lceil v_j - 1 \rceil = \tilde{v}_j \qquad (\; v_j^* = \lceil v_j - \log(v_j + 1) \rceil = \tilde{v}_j \;).$$

Therefore P also satisfies $(\tilde{H}, \tilde{V})$. □

Lemma 2. *If there exists a polyomino P with projections $(\tilde{H}, \tilde{V})$, then there exists a polyomino with row projections $H^* \in \{1, \ldots, n\}^m$ and column projections $V^* \in \{1, \ldots, m\}^n$, such that (H^*, V^*) is the approximation with the absolute (logarithmic) error of (H, V).*

Proof. From definition of (H, V) (for both kinds of approximation) we have that every component of (H, V) can be rounded to the corresponding component of $(\tilde{H}, \tilde{V})$. Hence P is also realisation of (H, V). □

Because we know that RPfOP for polyominoes, h-convex polyominoes and v-convex polyominoes is NP-complete (see [1] and [8]) we obtain from Lemma 1 and Lemma 2 the following result

Theorem 1. *The reconstruction of polyominoes, h-convex polyominoes and v-convex polyominoes from their approximately orthogonal projections is NP-complete.* □

3 hv-Convex Polyomino

In this section we use some ideas and notations from Chrobak and Dürr [3] while describing the algorithm for reconstruction hv-convex polyominoes from approximately orthogonal projection.

In the algorithm described below we generalise the error of approximation and assume that it is in the form of a function f. We assume that the function f is positive on $\mathbb{R}_+$. For example, the absolute error is a constant function equal 1 ($f(x) = 1$) and the logarithmic error is a logarithmic function ($f(x) = \log(x+1)$).

First we define some auxiliary expressions:

$$\check{v}_j = \max\{1, \lceil v_j - f(v_j) \rceil\} \quad \text{and} \quad \hat{v}_j = \min\{m, \lfloor v_j + f(v_j) \rfloor\}$$

for $j = 1, \ldots, n$, and

$$\check{h}_i = \max\{1, \lceil h_i - f(h_i) \rceil\} \quad \text{and} \quad \hat{h}_i = \min\{n, \lfloor h_i + f(h_i) \rfloor\}$$

for $i = 1, \ldots, m$.

These expressions have following properties: $\check{h}_i$ ($\check{v}_j$) is the minimal positive integer value and $\hat{h}_i$ ($\hat{v}_j$) is the maximal positive integer value for an horizontal (vertical) projection that differs by at most $f(h_i)$ ($f(v_j)$) from h_i (v_j).

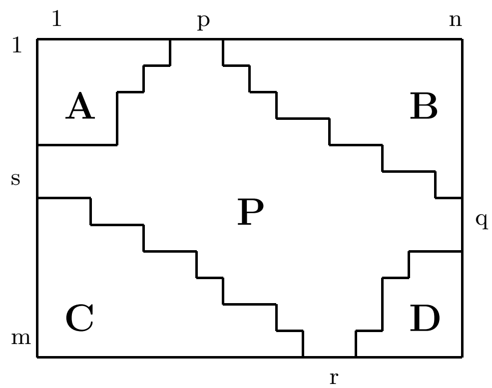

Fig. 1. The convex polyomino P that is anchored at (p, q, r, s), with corner regions A, B, C and D

Let (i, j) denote the cell of matrix that is in the i-th row and the j-th column. For an hv-convex polyomino an object A is called an *upper-left corner region* (see Fig.1) if $(i + 1, j) \in A$ or $(i, j + 1) \in A$ implies $(i, j) \in A$. In an analogous way we can define other corner regions. By $\overline{P}$ we denote the complement of P.

From the definition of hv-convex polyominoes we have the following lemma

Lemma 3 (Chrobak and Dürr [3]). *P is an hv-convex polyomino if and only if*

$$P = \overline{A \cup B \cup C \cup D},$$

where A, B, C, D are disjoint corner regions (upper-left, upper-right, lower-left and lower-right, respectively) such that

(i) $(i - 1, j - 1) \in A$ *implies* $(i, j) \notin D$, *and*

(ii) $(i - 1, j + 1) \in B$ *implies* $(i, j) \notin C$. □

We say that the hv-convex polyomino P is anchored at (p, q, r, s) if cells $(1, p), (q, n), (m, r), (s, 1) \in P$ (i.e. these cells do not belong to any corner region).

The main idea of our algorithm is, given (H,V) – vectors of approximately orthogonal projections, to construct a 2SAT expression $F_{p,q,r,s}(H, V)$ with the property that $F_{p,q,r,s}(H, V)$ is satisfiable if and only if there exists an hv-convex polyomino with projections (H^*, V^*) that is anchored at (p, q, r, s) and every component of (H, V) differs by at most the value of function f from the corresponding component of (H^*, V^*).

$F_{p,q,r,s}(H, V)$ consists of several sets of clauses, each set expressing a certain property: “Corners” (Cor), “Connectivity” (Con), “Anchors” (Anc), “Lower bound on column sums” (LBC), “Upper bound on column sums” (UBC), “Lower bound on row sums” (LBR) and “Upper bound on row sums” (UBR).

$$Cor \equiv \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow A_{i-1,j} & A_{i,j} \Rightarrow A_{i,j-1} \\ B_{i,j} \Rightarrow B_{i-1,j} & B_{i,j} \Rightarrow B_{i,j+1} \\ C_{i,j} \Rightarrow C_{i+1,j} & C_{i,j} \Rightarrow C_{i,j-1} \\ D_{i,j} \Rightarrow D_{i+1,j} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

$$Con \equiv \bigwedge_{i,j} \{ A_{i,j} \Rightarrow \overline{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \overline{C}_{i+1,j-1} \}$$

$$Anc_{p,q,r,s} \equiv \bigwedge \left\{ \begin{array}{llll} \overline{A}_{1,p} & \overline{B}_{1,p} & \overline{C}_{1,p} & \overline{D}_{1,p} \\ \overline{A}_{q,n} & \overline{B}_{q,n} & \overline{C}_{q,n} & \overline{D}_{q,n} \\ \overline{A}_{m,r} & \overline{B}_{m,r} & \overline{C}_{m,r} & \overline{D}_{m,r} \\ \overline{A}_{s,1} & \overline{B}_{s,1} & \overline{C}_{s,1} & \overline{D}_{s,1} \end{array} \right\}$$

$$LBC \equiv \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow \overline{C}_{i+\check{v}_j,j} & A_{i,j} \Rightarrow \overline{D}_{i+\check{v}_j,j} \\ B_{i,j} \Rightarrow \overline{C}_{i+\check{v}_j,j} & B_{i,j} \Rightarrow \overline{D}_{i+\check{v}_j,j} \end{array} \right\} \wedge \bigwedge_{j} \left\{ \begin{array}{l} \overline{C}_{\check{v}_j,j} \\ \overline{D}_{\check{v}_j,j} \end{array} \right\}$$

$$UBC_{p,r} \equiv \bigwedge_{i} \left\{ \begin{array}{l} \bigwedge_{j \le \min\{p,r\}} \overline{A}_{i,j} \Rightarrow C_{i+\hat{v}_j,j} \\ \bigwedge_{p \le j \le r} \overline{B}_{i,j} \Rightarrow C_{i+\hat{v}_j,j} \\ \bigwedge_{r \le j \le p} \overline{A}_{i,j} \Rightarrow D_{i+\hat{v}_j,j} \\ \bigwedge_{\max\{p,r\} \le j} \overline{B}_{i,j} \Rightarrow D_{i+\hat{v}_j,j} \end{array} \right\}$$

$$LBR \equiv \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow \overline{B}_{i,j+\check{h}_i} & A_{i,j} \Rightarrow \overline{D}_{i,j+\check{h}_i} \\ C_{i,j} \Rightarrow \overline{B}_{i,j+\check{h}_i} & C_{i,j} \Rightarrow \overline{D}_{i,j+\check{h}_i} \end{array} \right\} \wedge \bigwedge_{i} \left\{ \begin{array}{l} \overline{B}_{i,\check{h}_i} \\ \overline{D}_{\check{h}_i} \end{array} \right\}$$

$$UBR_{s,q} \equiv \bigwedge_{j} \left\{ \begin{array}{l} \bigwedge_{i \le \min\{s,q\}} \overline{A}_{i,j} \Rightarrow B_{i,j+\hat{h}_i} \\ \bigwedge_{s \le i \le q} \overline{C}_{i,j} \Rightarrow B_{i,j+\hat{h}_i} \\ \bigwedge_{q \le j \le s} \overline{A}_{i,j} \Rightarrow D_{i,j+\hat{h}_i} \\ \bigwedge_{\max\{s,q\} \le j} \overline{C}_{i,j} \Rightarrow D_{i,j+\hat{h}_i} \end{array} \right\}$$

(LBC) assigns the minimal distance between corner regions for columns (for j-th column it is equal to $\check{v}_j$). (UBC) assigns the maximal distance between corner regions for columns (for j-th column it is equal to $\hat{v}_j$). (LBR) and (UBR) are analogous for rows. Now we define a 2SAT formula

$$F_{p,q,r,s}(H,V) = Cor \wedge Con \wedge Anc_{p,q,r,s} \wedge LBC \wedge UBC_{p,r} \wedge LBR \wedge UBR_{q,s}.$$

All literals with indices outside the set $\{1,\ldots,m\} \times \{1,\ldots,n\}$ are assumed to have value 1.

Now we give the algorithm of reconstruction.

Algorithm

Input: $H \in \mathbb{R}_+^m$, $V \in \mathbb{R}_+^n$

`FOR` $p, r = 1, \ldots, n$ `AND` $q, s = 1, \ldots, m$ `DO`
`IF` $F_{p,q,r,s}(H,V)$ is satisfiable
`THEN RETURN` $P = \overline{A \cup B \cup C \cup D}$ `AND HALT`
`RETURN` "NO"

Theorem 2. *$F_{p,q,r,s}(H,V)$ is satisfiable if and only if P is an hv-convex polyomino with projections (H^*, V^*) that is anchored at (p,q,r,s) and every component of (H^*, V^*) differs from the correspondent component of (H,V) by at most the value of function f for this component.*

Proof. ($\Leftarrow$) If P is an hv-convex polyomino with properties like in the theorem, then let A, B, C, D be the corner regions from Lemma 3. By Lemma 3, A, B, C, D satisfy conditions (Cor) and (Con). Condition (Anc) is true because P is anchored at (p,q,r,s). Moreover for all $i = 1, \ldots, m$ we have $|h_i^* - h_i| \le f(h_i)$ and $h_i^* \in \mathbb{N}$, hence $\check{h}_i \le h_i^* \le \hat{h}_i$ and conditions (LBR) and (UBR) hold. Analogous, conditions (LBC) and (UBC) hold for vertical projections.

($\Rightarrow$) If $F_{p,q,r,s}(H,V)$ is satisfiable, take $P = \overline{A \cup B \cup C \cup D}$. Conditions (Cor), (Con), (LBC) and (LBR) imply that the sets A, B, C, D satisfy Lemma 3 ((LBC) and (LBR) guarantee disjointness of A, B, C, D), and thus P is an hv-convex polyomino. Also, by (Anc), P is anchored at (p,q,r,s). Conditions (LBR) and (UBR) imply that $\check{h}_i \le h_i^* \le \hat{h}_i$ for each row i. Hence $|h_i^* - h_i| \le f(h_i)$. Analogous, conditions (LBC) and (UBC) imply that $\check{v}_j \le v_j^* \le \hat{v}_j$ for each column j and therefore $|v_j^* - v_j| \le f(v_j)$. Moreover because P is the finite set we have $\sum_i h_i^* = \sum_j v_j^*$. Hence P must be an hv-convex polyomino with approximately orthogonal projections (H,V) with respect to function f. $\square$

Each formula $F_{p,q,r,s}(H,V)$ has size $O(mn)$ and can be computed in the linear time. Since a 2SAT formula can also be solved in the linear time, we obtain the following result

Theorem 3. *The problem of reconstruction of hv-convex polyominoes from approximately orthogonal projections can be solved in time $O(m^3n^3)$.* $\square$

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