Note that for $Z \neq 0$ we can transfer (2) back to (1). If the triple (X, Y, Z) satisfies the equation then it is satisfied also by $(\lambda X, \lambda Y, \lambda Z)$ for any $\lambda \in K$:

$$\lambda^3 Y^2 Z = \lambda^3 X^3 + \lambda^3 a X Z^2 + b \lambda^3 Z^3$$

$$(\lambda Z)^2 (\lambda Z) = (\lambda X)^3 + a (\lambda X)(\lambda Z)^2 + b(\lambda Z)^3$$

Of course point $(0,0,0)$ satisfies (2) for any a, b. So this particular point is not interesting and we exclude it from further considerations.

Projective plane:

By projective plane we mean the set of equivalence classes of triples (X, Y, Z), such that not all coordinates are equal to zero, where two triples

$$(X_1, Y_1, Z_1), (X_2, Y_2, Z_2)$$

are said to be equivalent if they are a scalar multiple of one another, i.e.:

$$\exists \lambda \in K \setminus \{0\} \quad (X_1, Y_1, Z_1) = (\lambda X_2, \lambda Y_2, \lambda Z_2)$$

$\lambda \in K \setminus \{0\}$: $(0,0,0)$ is not "interesting"

Such a straight line $\lambda(X, Y, Z)$ without $(0,0,0)$ point we call a projective point.

If a projective point has $Z \neq 0$ then there is exactly one triple of the form $(x, y, 1)$ in the equivalence class of the point.

For a point of E having $Z=0$ we get from (2) that $X=0$, and since $(X,Y,Z)\neq(0,0,0)$ we get that $Y\neq 0$. As a representative of the class $(0:Y:0)$ we take $(0,1,0)$.

$(0:Y:0)$ — class of abstraction

The class is the "zero at infinity" $\mathcal{O}$.

As we see $\mathcal{O}$ is naturally expressed in projective coordinates.

Let us derive the point addition formulas in projective coordinates:

As a start point for the derivation we shall use formulas for affine coordinates.

Point doubling in projective coordinates:

We have

$$\lambda = \frac{3x_1^2+a}{2y_1} = \frac{3(X_1/Z_1)^2+a}{2(Y_1/Z_1)} =$$

$$= \frac{3(X_1/Z_1)^2+a}{2(Y_1/Z_1)} \cdot \frac{Z_1^2}{Z_1^2} = \frac{3X_1^2+aZ_1^2}{2Y_1Z_1}$$

Assign:

$$W := 3X_1^2+aZ_1^2$$

$$S := 2Y_1Z_1$$

Thus $\lambda = \frac{W}{S}$

$$x_3 = \lambda^2 - 2x_1 = \left(\frac{W}{S}\right)^2 - 2\left(\frac{X_1}{Z_1}\right) =$$

$$= \frac{W^2}{S^2} - \frac{2X_1}{Z_1} \cdot \frac{4Y_1^2Z_1}{4Y_1^2Z_1} = \frac{W^2}{S^2} - \frac{8X_1Y_1^2Z_1}{4Y_1^2Z_1^2} =$$

$$= \frac{W^2}{S^2} - \frac{4SX_1Y_1}{S^2} = \frac{W^2-4SX_1Y_1}{S^2}$$

Assign: $B := 2SX_1Y_1$ $\quad h := W^2-2B$

So $x_3 = \frac{h}{S^2}$

$$y_3 = \lambda(x_1 - x_3) - y_1 = \frac{W}{S} \cdot \left(\frac{X_1}{Z_1} - \frac{h}{S^2}\right) - \frac{Y_1}{Z_1} =$$

$$= \frac{W}{S} \cdot \left(\frac{X_1}{Z_1} \cdot \frac{4Y_1^2 Z_1}{4Y_1^2 Z_1} - \frac{h}{S^2}\right) - \frac{Y_1}{Z_1} =$$

$$= \frac{W}{S} \left(\frac{4X_1 Y_1^2 Z_1}{4Y_1^2 Z_1^2} - \frac{h}{S^2}\right) - \frac{Y_1}{Z_1} =$$

$$= \frac{W}{S} \left(\frac{B}{S^2} - \frac{h}{S^2}\right) - \frac{Y_1}{Z_1} \cdot \frac{8Y_1^3 Z_1^2}{8Y_1^3 Z_1^2} =$$

$$= \frac{W(B-h)}{S^3} - \frac{8Y_1^4 Z_1^2}{8Y_1^3 Z_1^3} = \frac{W(B-h)}{S^3} - \frac{2 \cdot S^2 Y_1^2}{S^3} =$$

$$= \frac{W(B-h) - 2(SY_1)^2}{S^3}$$

If we adjust denominator in x_3:

$$x_3 = \frac{hS}{S^3}$$

Then we can define:

$$\begin{cases} x_3 = hS \\ y_3 = W(B-h) - 2(SY_1)^2 \\ z_3 = S^3 \\ W = 3X_1^2 + aZ_1^2 \\ S = 2Y_1 Z_1 \\ B = 2SX_1 Y_1 \\ h = W^2 - 2B \end{cases}$$

Point addition in projective coordinates:

See that the affine test $x_1 = x_2$ corresponds to

$$\frac{X_1}{Z_1} = \frac{X_2}{Z_2}$$

$$X_1 Z_2 = X_2 Z_1 \quad ?$$

So $x_1 \neq x_2$ means that $X_1 Z_2 \neq X_2 Z_1$.

We have $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ used in the affine point addition formula.

$$\mathcal{L} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2/z_2 - y_1/z_1}{x_2/z_2 - x_1/z_1} \cdot \frac{z_1 z_2}{z_1 z_2} =$$

$$= \frac{y_2 z_1 - y_1 z_2}{x_2 z_1 - x_1 z_2}$$

Define:

$$u_1 := x_1 z_2$$

$$u_2 := x_2 z_1$$

$$s_1 := y_1 z_2$$

$$s_2 := y_2 z_1$$

$$p := u_2 - u_1$$

$$R := s_2 - s_1$$

Then

$$\mathcal{L} = \frac{s_2 - s_1}{u_2 - u_1} = \frac{R}{p}$$

$$x_3 = \mathcal{L}^2 - x_1 - x_2 = \left(\frac{R}{p}\right)^2 - \frac{x_1}{z_1} - \frac{x_2}{z_2}$$

$$= \frac{R^2}{p^2} \cdot \frac{z_1 z_2}{z_1 z_2} - \frac{p^2 z_2 x_1}{p^2 z_1 z_2} - \frac{p^2 z_1 x_2}{p^2 z_1 z_2} =$$

$$= \frac{R^2 \cdot z_1 \cdot z_2 - p^2 (u_1 + u_2)}{p^2 z_1 z_2}$$

Define: $w := z_1 \cdot z_2$

$$x_3 := \frac{R^2 w - p^2 (u_1 + u_2)}{p^2 \cdot w}$$

$$y_3 = \mathcal{L}(x_1 - x_3) - y_1 = \frac{R}{p}\left(\frac{x_1}{z_1} - \frac{R^2 w - p^2(u_1+u_2)}{p^2 w}\right) - \frac{y_1}{z_1} =$$

$$= \frac{R}{p}\left(\frac{x_1 \cdot z_2 \cdot p^2}{p^2 z_1 z_2} - \frac{R^2 w - p^2(u_1+u_2)}{p^2 \cdot w}\right) - \frac{y_1 z_2 p^3}{z_1 z_2 p^3} =$$

$$= R\left(\frac{u_1 \cdot p^2}{p^3 w} - \frac{R^2 w - p^2(u_1+u_2)}{p^3 w}\right) - \frac{s_1 \cdot p^3}{p^3 w} =$$

$$= \frac{1}{p^3 w}\left[R\left(u_1 p^2 - R^2 w + p^2(u_1+u_2)\right) - s_1 \cdot p^3\right]$$

Note that alternatively

$$y_3 = \underline{-(\beta + \alpha x_3)} = -(y_2 - \alpha x_2 + \alpha x_3) =$$

↑ the point opposite to the third point of the intersection of the straight line and the curve:

$$\beta = y_1 - \alpha x_1$$

or $\underline{\beta = y_2 - \alpha x_2}$

$$= -y_2 + \alpha(x_2 - x_3)$$

And the alternative representation can be directly translated to:

$$y_3 = \frac{1}{p^3 W}\left[R\left(U_2 \cdot P^2 - R^2 W + P^2(U_1 + U_2)\right) - S_2 P^3\right]$$

Adding together both representations of the same y_3 we get:

$$2y_3 = \frac{1}{p^3 W}\left[R\left((U_1 + U_2)P^2 - 2R^2 W + 2P^2(U_1 + U_2)\right) - (S_1 + S_2)P^3\right] =$$

$$\frac{1}{p^3 W}\left[R\left(-2WR^2 + 3(U_1 + U_2)P^2\right) - (S_1 + S_2)\cdot P^3\right]$$

if we adjust the denominator in x_3:

$$x_3 = \frac{P\left(WR^2 - (U_1 + U_2)\cdot P^2\right)}{p^3 W}$$

then we can define:

$$\begin{cases} x_3 := P\left(WR^2 - (U_1 + U_2)P^2\right) \\ y_3 := \left[R\cdot\left(-2WR^2 + 3(U_1 + U_2)\cdot P^2\right) - (S_1 + S_2)\cdot P^3\right]/2 \\ z_3 := P^3 W \\ W := Z_1 Z_2 \\ P = U_2 - U_1 \\ R = S_2 - S_1 \\ U_1 = X_1 Z_2 \\ U_2 = X_2 Z_1 \\ S_1 = Y_1 Z_2 \\ S_2 = Y_2 Z_1 \end{cases}$$