TOPOLOGICALLY INVARIANT σ-IDEALS ON THE HILBERT CUBE

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Abstract. We study and classify topologically invariant σ-ideals with a Borel base on the Hilbert cube and evaluate their cardinal characteristics. One of the results of this paper solves (positively) a known problem whether the minimal cardinalities of the families of Cantor sets covering the unit interval and the Hilbert cube are the same.

1. Introduction and survey of principal results

In this paper we study properties of topologically invariant σ-ideals with Borel base on the Hilbert cube $\mathbb{I}^\omega = [0,1]^\omega$. In particular, we evaluate the cardinal characteristics of such σ-ideals. One of the results of this paper solves (positively) a known problem whether the minimal cardinalities of the families of Cantor sets covering the unit interval and the Hilbert cube are the same.

Topologically invariant ideals in the finite dimensional case were considered in paper [3] devoted to studying topologically invariant σ-ideals with Borel base on Euclidean spaces $\mathbb{R}^n$. To present the principal results, we need to recall some definitions.

A non-empty family $\mathcal{I}$ of subsets of a set X is called an *ideal on* X if $\mathcal{I}$ is hereditary with respect to taking subsets and $\mathcal{I}$ is additive (in the sense that $A \cup B \in \mathcal{I}$ for any subsets $A, B \in \mathcal{I}$). An ideal $\mathcal{I}$ on X is called a *σ-ideal* if for each countable subfamily $\mathcal{A} \subseteq \mathcal{I}$ the union $\bigcup \mathcal{A}$ belongs to $\mathcal{I}$. An ideal $\mathcal{I}$ on X will be called *non-trivial* if $\mathcal{I}$ contains some uncountable subset of X and $\mathcal{I}$ does not coincide with the ideal $\mathcal{P}(X)$ of all subsets of X. Each family $\mathcal{F}$ of subsets of a set X generates the σ-ideal $\sigma\mathcal{F}$ consisting of subsets of countable unions of sets from the family $\mathcal{F}$.

A subset A of a topological space X has *the Baire property* (briefly, is a *BP-set*) if there is an open set $U \subseteq X$ such that the symmetric difference $A \triangle U = (A \setminus U) \cup (U \setminus A)$ is meager in X (i.e., is a countable union of nowhere dense subsets of X). A subfamily $\mathcal{B} \subseteq \mathcal{I}$ is called a *base* for $\mathcal{I}$ if each set $A \in \mathcal{I}$ is contained in some set $B \in \mathcal{B}$. We shall say that an ideal $\mathcal{I}$ on a Polish space X has *σ-compact base* (resp. *Borel base*, *analytic base*, *BP-base*) if $\mathcal{I}$ has a base consisting of σ-compact (resp. Borel, analytic, BP-) subsets of X. Let us recall that a subset A of a Polish space X is *analytic* if A is the image of a Polish space under a continuous map. It is well-known that each Borel subset of a Polish space X is analytic and each analytic subset of X has the Baire property. Thus, for an ideal $\mathcal{I}$ on a Polish space X we have the following implications:

$$\mathcal{I} \text{ has } \sigma\text{-compact base} \Rightarrow \mathcal{I} \text{ has Borel base} \Rightarrow \mathcal{I} \text{ has analytic base} \Rightarrow \mathcal{I} \text{ has BP-base.}$$

Classical examples of σ-ideals with Borel base on the real line $\mathbb{R}$ are the ideal $\mathcal{M}$ of meager subsets and the ideal $\mathcal{N}$ of Lebesgue null subsets of $\mathbb{R}$. One of the differences between these ideals is that the ideal $\mathcal{M}$ is topologically invariant while $\mathcal{N}$ is not.

We shall say that an ideal $\mathcal{I}$ on a topological space X is *topologically invariant* if $\mathcal{I}$ is preserved by homeomorphisms of X in the sense that $\mathcal{I} = \{h(A) : A \in \mathcal{I}\}$ for each homeomorphism $h : X \to X$ of X.

In [3] we proved that the ideal $\mathcal{M}$ of meager subsets of a Euclidean space $\mathbb{R}^n$ is the largest topologically invariant σ-ideal with BP-base on $\mathbb{R}^n$. This is not true anymore for the Hilbert cube $\mathbb{I}^\omega$ as shown by the σ-ideal $\sigma\mathcal{D}_0$ of countable-dimensional subsets of $\mathbb{I}^\omega$. The σ-ideal $\sigma\mathcal{D}_0$ is generated by all zero-dimensional subspaces of $\mathbb{I}^\omega$ and has a base consisting of countable-dimensional $G_{\delta\sigma}$-sets. It is clear that $\sigma\mathcal{D}_0 \not\subseteq \mathcal{M}$. So, $\mathcal{M}$ is not the largest non-trivial σ-ideal with Borel base on $\mathbb{I}^\omega$. Nonetheless, the ideal $\mathcal{M}$ has the following maximality property.

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Theorem 1.1. *The ideal $\mathcal{M}$ of meager subsets of the Hilbert cube $\mathbb{I}^\omega$ is:*

(1) *a maximal non-trivial topologically invariant ideal with BP-base on $\mathbb{I}^\omega$, and*
(2) *the largest non-trivial topologically invariant ideal with σ-compact base on $\mathbb{I}^\omega$.*

Proof. (1) Given a non-trivial topologically invariant ideal $\mathcal{I} \supseteq \mathcal{M}$ with BP-base on $\mathbb{I}^\omega$ we should prove that $\mathcal{I} = \mathcal{M}$. Assume that $\mathcal{I}$ contains some subset $A \in \mathcal{I} \setminus \mathcal{M}$. Since $\mathcal{I}$ has BP-base, we can additionally assume that the set A has the Baire property in $\mathbb{I}^\omega$. Being non-meager, the BP-set A contains a G_δ-subset $G_U \subseteq A$, dense in some non-empty open set $U \subseteq Q$. The compactness and the topological homogeneity of the Hilbert cube (see e.g. [17, 6.1.6]) allow us to find a finite sequence of homeomorphisms $h_1, \dots, h_n : \mathbb{I}^\omega \to \mathbb{I}^\omega$ such that $\mathbb{I}^\omega = \bigcup_{i=1}^n h_i(U)$. The topological invariance of the ideal $\mathcal{I}$ guarantees that the dense G_δ-set $G = \bigcup_{i=1}^n G_U$ belongs to the ideal $\mathcal{I}$. Since $\mathbb{I}^\omega \setminus G \in \mathcal{M} \subseteq \mathcal{I}$ is meager, we conclude that $\mathbb{I}^\omega = G \cup (\mathbb{I}^\omega \setminus G) \in \mathcal{I}$, which means that the ideal $\mathcal{I}$ is trivial.

(2) Next, assume that $\mathcal{I}$ is a non-trivial topologically invariant ideal on $\mathbb{I}^\omega$ with σ-compact base. To show that $\mathcal{I} \subseteq \mathcal{M}$, it suffices to check that each σ-compact set $K \in \mathcal{I}$ is meager in $\mathbb{I}^\omega$. Assuming that K is not meager and applying Baire Theorem, we conclude that K contains a non-empty open subset $U \subseteq \mathbb{I}^\omega$. By the compactness and the topological homogeneity of $\mathbb{I}^\omega$ there are homeomorphisms $h_1, \dots, h_n$ of $\mathbb{I}^\omega$ such that $\mathbb{I}^\omega = \bigcup_{i=1}^n h_i(U)$. Now the topological invariance and the additivity of $\mathcal{I}$ imply that $\mathbb{I}^\omega \in \mathcal{I}$, which means that the ideal $\mathcal{I}$ is trivial. □

In [3] we proved that the family of all non-trivial topologically invariant σ-ideals with analytic base on a Euclidean space $\mathbb{R}^n$ contains the smallest element, namely the σ-ideal $\sigma\mathcal{C}_0$ generated by so called tame Cantor sets in $\mathbb{R}^n$. A similar fact holds also for topologically invariant ideals with an analytic base on the Hilbert cube $\mathbb{I}^\omega$.

By a *Cantor set* in $\mathbb{I}^\omega$ we understand any subset $C \subseteq \mathbb{I}^\omega$ homeomorphic to the Cantor cube $\{0,1\}^\omega$. By Brouwer's characterization [15] of the Cantor cube, a closed subset $C \subseteq \mathbb{I}^\omega$ is a Cantor set if and only if C is zero-dimensional and has no isolated points. A Cantor set $A \subseteq \mathbb{I}^\omega$ is called *minimal* if for each Cantor set $B \subseteq \mathbb{I}^\omega$ there is a homeomorphism $h : \mathbb{I}^\omega \to \mathbb{I}^\omega$ such that $h(A) \subseteq B$.

Minimal Cantor sets in the Hilbert cube $\mathbb{I}^\omega$ can be characterized as Cantor Z_ω-sets. Let us recall that a closed subset A of a topological space X is called a *Z_n-set* in X for $n \le \omega$ if the set $\{f \in C(\mathbb{I}^n, X) : f(\mathbb{I}^n) \cap A = \emptyset\}$ is dense in the space $C(\mathbb{I}^n, X)$ of all continuous functions from $\mathbb{I}^n$ to X, endowed with the compact-open topology. A closed subset A of a topological space X is called a *Z-set* in X if for any open cover $\mathcal{U}$ of X there is a continuous map $f : X \to X \setminus A$, which is *$\mathcal{U}$-near* to the identity map in the sense that for each $x \in X$ the set $\{f(x), x\}$ is contained in some set $U \in \mathcal{U}$. It is clear that a subset of the Hilbert cube is a Z-set in $\mathbb{I}^\omega$ if and only if it is a Z_ω-set in $\mathbb{I}^\omega$. By the Z-Set Unknotting Theorem 11.1 in [8], any two Cantor Z-sets $A, B \subseteq \mathbb{I}^\omega$ are *abmiently homeomorphic*, which means that there is a homeomorphism $h : \mathbb{I}^\omega \to \mathbb{I}^\omega$ such that $h(A) = B$. This implies that a Cantor set $C \subseteq \mathbb{I}^\omega$ is minimal if and only if it is a Z-set in $\mathbb{I}^\omega$.

By $\mathcal{C}_0$ we denote the family of all minimal Cantor sets in $\mathbb{I}^\omega$ and by $\sigma\mathcal{C}_0$ the σ-ideal generated by the family $\mathcal{C}_0$. Observe that $\sigma\mathcal{C}_0$ coincides with the σ-ideal generated by zero-dimensional Z-sets in $\mathbb{I}^\omega$. The following theorem shows that $\sigma\mathcal{C}_0$ is the smallest non-trivial σ-ideal with analytic base on $\mathbb{I}^\omega$.

Theorem 1.2. *The family $\mathcal{C}_0$ (the σ-ideal $\sigma\mathcal{C}_0$) is contained in each topologically invariant non-trivial (σ-)ideal $\mathcal{I}$ with analytic base on $\mathbb{I}^\omega$.*

Proof. Let $C \in \mathcal{C}_0$. By the hypothesis there exists an uncountable analytic set $A \in \mathcal{I}$. It is known that A contains a copy C_1 of a Cantor set. By the minimality of C there exists a homeomorphism h of the Hilbert cube such that $h(C) \subseteq C_1 \subseteq A$. Hence $h(C) \in \mathcal{I}$. By the topological invariance of $\mathcal{I}$ also $C \in \mathcal{I}$. □

As it is known, on the Hilbert cube there are non-trivial topologically invariant σ-ideals with Borel base, which are not contained in the ideal $\mathcal{M}$ of meager sets. It turns out that among such σ-ideals there is the smallest one. It is denoted by $\sigma\mathcal{G}_0$ and is generated by minimal dense G_δ-subsets of $\mathbb{I}^\omega$.

A dense G_δ-subset A of a Polish space X will be called *minimal* if for each dense G_δ-set $B \subseteq X$ there is a homeomorphism $h : X \to X$ such that $h(A) \subseteq B$. By [4], any two minimal dense G_δ-subsets of $\mathbb{I}^\omega$ are ambiently homeomorphic. Minimal dense G_δ-sets in $\mathbb{I}^\omega$ were characterized in [4] as dense tame G_δ-set. To introduce tame G_δ-sets in the Hilbert cube we need some additional notions.

A family $\mathcal{V}$ of subsets of a topological space X is called *vanishing* if for any open cover $\mathcal{U}$ of X the subfamily $\{V \in \mathcal{V} : \forall U \in \mathcal{U}\ \ V \not\subseteq U\}$ is locally finite in X.

An open subset U of $\mathbb{I}^\omega$ is called a *tame open ball* if

- its closure $\bar{U}$ in $\mathbb{I}^\omega$ is homeomorphic to the Hilbert cube;
- its boundary ∂U in $\mathbb{I}^\omega$ is homeomorphic to the Hilbert cube;
- ∂U is a Z-set in $\bar{U}$ and in $\mathbb{I}^\omega \setminus U$.

By [8, 12.2], tame open balls form a base of the topology of the Hilbert cube.

A subset U of $\mathbb{I}^\omega$ is called a *tame open set* in $\mathbb{I}^\omega$ if $U = \bigcup \mathcal{U}$ for some vanishing family $\mathcal{U}$ of tame open balls with pairwise disjoint closures in $\mathbb{I}^\omega$. The family $\mathcal{U}$ is unique and coincides with the family $\mathcal{C}(U)$ of all connected components of U. By $\bar{\mathcal{C}}(U) = \{\bar{C} : C \in \mathcal{C}(U)\}$ we shall denote the (disjoint) family of closures of the connected components of the set U.

A subset G of $\mathbb{I}^\omega$ is called a *tame G_δ-set* in $\mathbb{I}^\omega$ if $G = \bigcap_{n\in\omega} U_n$ for some sequence $(U_n)_{n\in\omega}$ of tame open sets in $\mathbb{I}^\omega$ such that $\bigcup \bar{\mathcal{C}}(U_{n+1}) \subseteq U_n$ for every $n \in \omega$ and the family $\bigcup_{n\in\omega} \bar{\mathcal{C}}(U_n)$ is vanishing in $\mathbb{I}^\omega$.

By Theorem 4 of [4], a dense G_δ-set in $\mathbb{I}^\omega$ is minimal if and only if it is dense tame G_δ in $\mathbb{I}^\omega$.

Denote by $\mathcal{G}_0$ the family of all minimal dense G_δ-sets in $\mathbb{I}^\omega$ and by $\sigma\mathcal{G}_0$ the σ-ideal generated by the family $\mathcal{G}_0$. It is clear that $\mathcal{G}_0 \not\subseteq \mathcal{M}$. It turns out that the σ-ideal $\sigma\mathcal{G}_0$ is the smallest topologically invariant σ-ideal with BP-base, which is not contained in the ideal $\mathcal{M}$ of meager subsets in $\mathbb{I}^\omega$.

Theorem 1.3. *The family $\mathcal{G}_0$ (the σ-ideal $\sigma\mathcal{G}_0$) is contained in each topologically invariant (σ-)ideal $\mathcal{I} \not\subseteq \mathcal{M}$ with BP-base on $\mathbb{I}^\omega$.*

Proof. If $\mathcal{I} \not\subseteq \mathcal{M}$, then we can find a non-meager set $A \in \mathcal{I}$. Repeating the argument from the proof of Theorem 1.1, we can show that the ideal $\mathcal{I}$ contains a dense G_δ-subset G of $\mathbb{I}^\omega$. To check that $\mathcal{G}_0 \subseteq \mathcal{I}$, fix any minimal dense G_δ-set $M \subseteq \mathbb{I}^\omega$ and find a homeomorphism h of $\mathbb{I}^\omega$ such that $h(M) \subseteq G \in \mathcal{I}$. Then $h(M) \in \mathcal{I}$ and $M \in \mathcal{I}$ by the topological invariance of $\mathcal{I}$. □

In light of Theorem 1.3, it is important to study the properties of the σ-ideal $\sigma\mathcal{G}_0$ and how it is related to other σ-ideals. Since each minimal dense G_δ-set in $\mathbb{I}^\omega$ is zero-dimensional, the ideal $\sigma\mathcal{G}_0$ is contained in the σ-ideal $\sigma\mathcal{D}_0$ generated by the family $\mathcal{D}_0$ of all zero-dimensional subspaces of $\mathbb{I}^\omega$. By [14, 1.5.8], the σ-ideal $\sigma\mathcal{D}_0$ contains the σ-ideal $\sigma\overline{\mathcal{D}}_\omega$ generated by the family $\overline{\mathcal{D}}_\omega$ of all closed finite-dimensional subsets of $\mathbb{I}^\omega$.

Theorem 1.4. *The family $\overline{\mathcal{D}}_\omega$ is contained in each topologically invariant ideal $\mathcal{I} \not\subseteq \mathcal{M}$ with BP-base on $\mathbb{I}^\omega$. Consequently, $\sigma\overline{\mathcal{D}}_\omega \subsetneq \sigma\mathcal{G}_0 \subseteq \sigma\mathcal{D}_0$.*

Theorem 1.4 will be proved in Section 3. Theorems 1.2 and 1.3 will help us to evaluate the cardinal characteristics of an arbitrary topologically invariant σ-ideal with analytic base on the Hilbert cube.

Given an ideal $\mathcal{I}$ on a set $X = \bigcup \mathcal{I} \notin \mathcal{I}$, we shall consider the following four cardinal characteristics of $\mathcal{I}$:

$$\begin{aligned}
\mathrm{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I},\ \ \textstyle\bigcup \mathcal{A} \notin \mathcal{I}\},\\
\mathrm{non}(\mathcal{I}) &= \min\{|A| : A \subseteq X,\ \ A \notin \mathcal{I}\},\\
\mathrm{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I},\ \textstyle\bigcup \mathcal{A} = X\},\\
\mathrm{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I},\ \ \forall B \in \mathcal{I}\ \ \exists A \in \mathcal{A}\ \ (B \subseteq A)\}.
\end{aligned}$$

In fact, these four cardinal characteristics can be expressed using the following two cardinal characteristics defined for any pair $\mathcal{I} \subseteq \mathcal{J}$ of ideals:

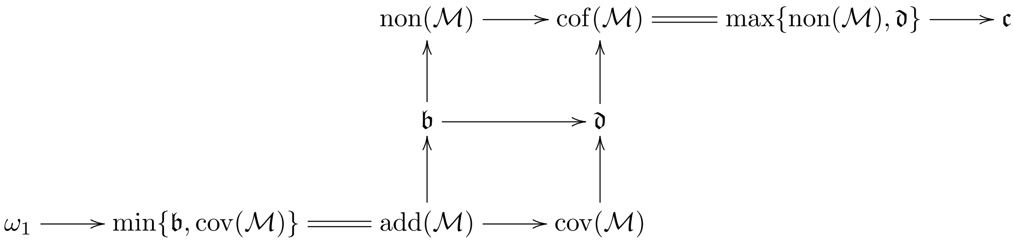
$$\begin{aligned}
\mathrm{add}(\mathcal{I}, \mathcal{J}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I},\ \ \textstyle\bigcup \mathcal{A} \notin \mathcal{J}\}\ \ \text{and}\\
\mathrm{cof}(\mathcal{I}, \mathcal{J}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J},\ \ \forall B \in \mathcal{I}\ \exists A \in \mathcal{A}\ \ (B \subseteq A)\}.
\end{aligned}$$

Namely,

$$\mathrm{add}(\mathcal{I}) = \mathrm{add}(\mathcal{I}, \mathcal{I}),\ \ \mathrm{non}(\mathcal{I}) = \mathrm{add}(\mathcal{F}, \mathcal{I}),\ \ \mathrm{cov}(\mathcal{I}) = \mathrm{cof}(\mathcal{F}, \mathcal{I}),\ \ \mathrm{cof}(\mathcal{I}) = \mathrm{cof}(\mathcal{I}, \mathcal{I})$$

where $\mathcal{F}$ stands for the ideal of finite subsets of X.

The cardinal characteristics of the σ-ideal $\mathcal{M}$ have been thoroughly studied in Set Theory, see [5] or [7]. They fit into the following (piece of Cichoń's) diagram in which an arrow $a \to b$ indicates that $a \le b$ in ZFC:



Here

$$\mathfrak{b} = \min\{|B| : B \subseteq \omega^\omega,\ \forall x \in \omega^\omega\ \exists y \in B\ \ (y \not\leq^* x)\} \text{ and}$$
$$\mathfrak{d} = \min\{|D| : D \subseteq \omega^\omega,\ \forall x \in \omega^\omega\ \exists y \in B\ \ (x \leq^* y)\}$$

are the *bounding* and *dominating* numbers [9], [18], [7] (the notation $x \leq^* y$ means $x(n) \leq y(n)$ for all but finitely many numbers n). The precise position of the small uncountable cardinals $\mathfrak{b}$ and $\mathfrak{d}$ in the interval $[\omega_1, \mathfrak{c}]$ depends on additional axioms of ZFC, see [5]. The same concerns the cardinal characteristics of the ideal $\mathcal{M}$ on the real line: their values depend on axioms, too, see [6].

The cardinal characteristics of the ideal $\mathcal{M}$ will be used to evaluate the cardinal characteristics of the σ-ideals $\sigma\mathcal{C}_0$ and $\sigma\mathcal{G}_0$ in the following theorems which are principal results of this paper. These two theorems will be proved in Sections 4 and 5, respectively.

Theorem 1.5. *The σ-ideal $\sigma\mathcal{C}_0$ on the Hilbert cube has cardinal characteristics:*

(1) $\mathrm{cov}(\sigma\mathcal{C}_0) = \mathrm{cov}(\mathcal{M})$,
(2) $\mathrm{non}(\sigma\mathcal{C}_0) = \mathrm{non}(\mathcal{M})$,
(3) $\mathrm{add}(\sigma\mathcal{C}_0, \mathcal{M}) = \mathrm{add}(\mathcal{M})$,
(4) $\mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M}) = \mathrm{cof}(\mathcal{M})$.

Theorem 1.6. *The σ-ideal $\sigma\mathcal{G}_0$ on the Hilbert cube has cardinal characteristics:*

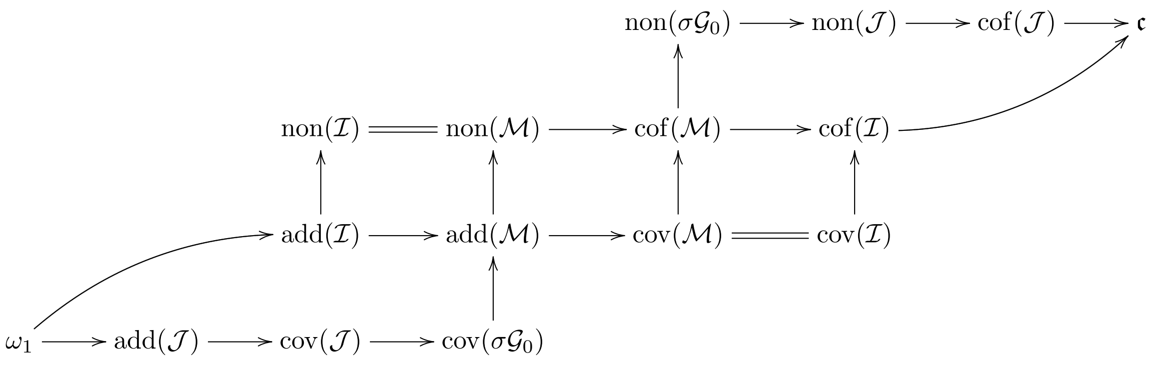
$$\omega_1 \leq \mathrm{add}(\sigma\mathcal{G}_0) \leq \mathrm{cov}(\sigma\mathcal{G}_0) \leq \mathrm{add}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{M}) \leq \mathrm{non}(\sigma\mathcal{G}_0) \leq \mathrm{cof}(\sigma\mathcal{G}_0) \leq \mathfrak{c}.$$

Theorems 1.2, 1.3, 1.5 and 1.6 imply the following corollary.

Corollary 1.7. *Let $\mathcal{I}$ be a non-trivial topologically invariant σ-ideal $\mathcal{I}$ with analytic base on the Hilbert cube.*

(1) *If* $\mathcal{I} \subseteq \mathcal{M}$, *then* $\mathrm{cov}(\mathcal{I}) = \mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{I}) = \mathrm{non}(\mathcal{M})$, $\mathrm{add}(\mathcal{I}) \leq \mathrm{add}(\mathcal{M})$, *and* $\mathrm{cof}(\mathcal{I}) \geq \mathrm{cof}(\mathcal{M})$.
(2) *If* $\mathcal{I} \not\subseteq \mathcal{M}$, *then* $\mathrm{add}(\mathcal{I}) \leq \mathrm{cov}(\mathcal{I}) \leq \mathrm{cov}(\sigma\mathcal{G}_0) \leq \mathrm{add}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{M}) \leq \mathrm{non}(\sigma\mathcal{G}_0) \leq \mathrm{non}(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I})$.

Corollary 1.7 implies that the cardinal characteristics of any non-trivial topologically invariant σ-ideals $\mathcal{I} \subseteq \mathcal{M}$ and $\mathcal{J} \not\subseteq \mathcal{M}$ with analytic base on the Hilbert cube fit in the following variant of Cichoń's diagram (in which $a \to b$ indicates that $a \leq b$):



The following example shows that the inequalities $\mathrm{add}(\mathcal{I}), \mathrm{cov}(\mathcal{J}) \leq \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{I}), \mathrm{non}(\mathcal{J})$ in this diagram can be strict. By an *arc* in $\mathbb{I}^\omega$ we understand any subset $A \subseteq \mathbb{I}^\omega$ homeomorphic to the closed interval $\mathbb{I} = [0, 1]$.

Example 1.8. (1) *The σ-ideal $\mathcal{I}$ generated by arcs in $\mathbb{I}^\omega$ has* $\mathrm{add}(\mathcal{I}) = \omega_1$, $\mathrm{cov}(\mathcal{I}) = \mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{I}) = \mathrm{non}(\mathcal{M})$ *and* $\mathrm{cof}(\mathcal{I}) = \mathfrak{c}$.

(2) *The ideal $\mathcal{J} = \sigma\mathcal{D}_0 \not\subseteq \mathcal{M}$ of countable dimensional subsets of $\mathbb{I}^\omega$ has* $\mathrm{add}(\mathcal{J}) = \mathrm{cov}(\mathcal{J}) = \omega_1$ *and* $\mathrm{non}(\mathcal{J}) = \mathrm{cof}(\mathcal{J}) = \mathfrak{c}$.

Proof. Using Corollary 1.7 to prove the two middle equalities, the first statement can be proved by analogy with Example 2.6 of [3]. To see that $\mathrm{non}(\sigma\mathcal{D}_0) = \mathfrak{c}$, observe that each subset $A \subseteq \mathbb{I}^\omega$ of cardinality $|A| < \mathfrak{c}$ is zero-dimensional. The equality $\mathrm{cov}(\sigma\mathcal{D}_0) = \omega_1$ is an old result of Smirnov, see [14, 5.1.B]. □

Next, we describe three classes of topologically invariant σ-ideals $\mathcal{I}$ with σ-compact base on the Hilbert cube whose cardinal characteristics coincide with the respective cardinal characteristics of the ideal $\mathcal{M}$. In the following definition by $\mathcal{H}(\mathbb{I}^\omega)$ we denote the (Polish) group of homeomorphisms of the Hilbert cube $\mathbb{I}^\omega$, endowed with the compact-open topology.

Definition 1.9. A topologically invariant ideal $\mathcal{I}$ on $\mathbb{I}^\omega$ is called

- *G_δ-generated* if $\mathcal{I}$ has σ-compact base and there is a G_δ-subset $G \subseteq \mathbb{I}^\omega$ such that each compact subset $K \subseteq G$ belongs to $\mathcal{I}$ and for each compact set $A \in \mathcal{I}$ the set $\mathcal{H}^G_A = \{h \in \mathcal{H}(\mathbb{I}^\omega) : h(A) \subseteq G\}$ is dense in $\mathcal{H}(\mathbb{I}^\omega)$;
- *G^*_δ-generated* if there is a G_δ-generated ideal $\mathcal{I}^*$ on $\mathbb{I}^\omega$ and an embedding $e : \mathbb{I}^\omega \to \mathbb{I}^\omega$ such that $\mathcal{I} = \{e^{-1}(A) : A \in \mathcal{I}^*\}$;
- *σG^*_δ-generated* if there is a sequence $(\mathcal{I}_n)_{n\in\omega}$ of G^*_δ-generated ideals on $\mathbb{I}^\omega$ such that $\bigcup_{n\in\omega} \mathcal{I}_n \subseteq \mathcal{I}$ and each $A \in \mathcal{I}$ is contained in the union $\bigcup_{n\in\omega} K_n$ of compact sets $K_n \in \mathcal{I}_n$, $n \in \omega$.

It follows that each G_δ-generated ideal is G^*_δ-generated, each G^*_δ-generated ideal is σG^*_δ-generated, and each σG^*_δ-generated σ-ideal has σ-compact base.

Theorem 1.10. *If a non-trivial topologically invariant σ-ideal $\mathcal{I}$ on $\mathbb{I}^\omega$ is σG^*_δ-generated, then*

$$\mathrm{add}(\mathcal{I}) = \mathrm{add}(\mathcal{M}),\ \ \mathrm{cov}(\mathcal{I}) = \mathrm{cov}(\mathcal{M}),\ \ \mathrm{non}(\mathcal{I}) = \mathrm{non}(\mathcal{M})\ \ \textit{and}\ \ \mathrm{cof}(\mathcal{I}) = \mathrm{cof}(\mathcal{M}).$$

Now we shall apply Theorem 1.10 to calculate the cardinal characteristics of some σ-ideals naturally appearing in Dimension Theory. We recall that for a family $\mathcal{F}$ of subsets of $\mathbb{I}^\omega$ by $\sigma\mathcal{F}$ we denote the smallest σ-ideal containing the family $\mathcal{F}$. It consists of all subsets of countable unions of sets from the family $\mathcal{F}$. By $\mathcal{K}$ we shall denote the family of all compact subsets of the Hilbert cube $\mathbb{I}^\omega$.

Definition 1.11. A family $\mathcal{D}$ of subsets of $\mathbb{I}^\omega$ will be called a *dimension class* if

- $\mathcal{D}$ is *compactly σ-additive* in the sense that the σ-ideal $\sigma(\mathcal{K} \cap \mathcal{D})$ is contained in the ideal $\mathcal{D}$;
- $\mathcal{D}$ is *topological* in the sense that a subset $A \subseteq \mathbb{I}^\omega$ belongs to the class $\mathcal{D}$ if it is homeomorphic to some set $B \in \mathcal{D}$;
- $\mathcal{D}$ contains a set $U \in \mathcal{D}$ which is *$\mathcal{K} \cap \mathcal{D}$-universal* in the sense that each compact subset $A \in \mathcal{D}$ is homeomorphic to a subspace of the universal space U;
- $\mathcal{D}$ *admits completions*, which means that each set $A \in \mathcal{D}$ is contained in a G_δ-subset $G \in \mathcal{D}$ of $\mathbb{I}^\omega$.

A family $\mathcal{D}$ of subsets of $\mathbb{I}^\omega$ is called a *σ-dimension class* if $\mathcal{D} = \bigcup_{n\in\omega} \mathcal{D}_n$ for an increasing sequence $(\mathcal{D}_n)_{n\in\omega}$ of dimension classes.

A typical example of a dimension class is the family $\dim_{\le n}$ of all subsets $A \subseteq \mathbb{I}^\omega$ of covering dimension $\dim(A) \le n$, see [14, §1.14]. The family $\dim_{<\omega}$ of all finite-dimensional subsets of $\mathbb{I}^\omega$ is an example of a σ-dimension class. More examples of (σ-)dimension classes can be found in Theory of Cohomological and Extension Dimensions [10], [11], [12], [13].

For every $n \in \omega$ denote by $\sigma\mathcal{Z}_n$ the σ-ideal on $\mathbb{I}^\omega$ generated by the family $\mathcal{Z}_n$ of all Z_n-sets in $\mathbb{I}^\omega$. Let us remark that the σ-ideal $\sigma\mathcal{Z}_0$ coincides with the ideal $\mathcal{M}$.

Theorem 1.12. (1) *For every $n \le \omega$ the σ-ideal $\sigma\mathcal{Z}_n$ is G_δ-generated.*

(2) *For each dimensional class $\mathcal{D}$ the σ-ideal $\sigma(\mathcal{Z}_\omega \cap \mathcal{D})$ is G_δ-generated and the σ-ideal $\sigma(\mathcal{K} \cap \mathcal{D})$ is G^*_δ-generated.*

(3) *For each σ-dimensional class $\mathcal{D}$ the σ-ideals $\sigma(\mathcal{Z}_\omega \cap \mathcal{D})$ and $\sigma(\mathcal{K} \cap \mathcal{D})$ are σG^*_δ-generated.*

Theorems 1.10 and 1.12 imply:

Corollary 1.13. *Each σ-ideal $\mathcal{I} \in \{\sigma\mathcal{Z}_n : n \in \omega\} \cup \{\sigma(\mathcal{K} \cap \mathcal{D}), \sigma(\mathcal{Z}_\omega \cap \mathcal{D}) : D$ is a σ-dimension class$\}$ has* $\mathrm{add}(\mathcal{I}) = \mathrm{add}(\mathcal{M})$, $\mathrm{cov}(\mathcal{I}) = \mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{I}) = \mathrm{non}(\mathcal{M})$, *and* $\mathrm{cof}(\mathcal{I}) = \mathrm{cof}(\mathcal{M})$.

This corollary answers Problem 2.6 of [2] concerning the cardinal characteristics of the σ-ideals $\sigma\mathcal{Z}_n$, $n \le \omega$, and $\sigma(\mathcal{K} \cap \dim_{\le 0})$.

2. Some Properties of Z-sets in the Hilbert cube

In this section we collect the necessary information on Z-sets in the Hilbert cube $\mathbb{I}^\omega$.

We recall that a subset A of a topological space X is called a *Z-set* in X if A is closed in X and for each open cover $\mathcal{U}$ of X there is a map $f : X \to X \setminus A$, which is $\mathcal{U}$-near to the identity map $\mathrm{id} : X \to X$. All *maps* considered in this paper *are assumed to be continuous.* In contrast, *functions need not be continuous.* It follows that a subset of the Hilbert cube $\mathbb{I}^\omega$ is a Z-set in $\mathbb{I}^\omega$ if and only if it is a Z_ω-set in $\mathbb{I}^\omega$.

A subset A of a topological space X is called a *σZ-set* in X is A can be written as the countable union $A = \bigcup_{n=1}^\infty A_n$ of Z-sets.

A typical Z-set in the Hilbert cube looks as in the following simple and known lemma.

Lemma 2.1. *For any closed proper subsets $A_n \subsetneqq \mathbb{I}$, $n \in \omega$, the product $\prod_{n\in\omega} A_n$ is a Z-set in $\mathbb{I}^\omega$.*

We shall often use the following powerful homogeneity property of the Hilbert cube [8, 11.1].

Theorem 2.2 (Z-Set Unknotting Theorem). *Any homeomorphism $h : A \to B$ between Z-sets A, B in the Hilbert cube $\mathbb{I}^\omega$ extends to a homeomorphism of $\mathbb{I}^\omega$.*

A map $f : X \to Y$ will be called a *Z-embedding* if $f(X)$ is a Z-set in Y and $f : X \to f(X)$ is a homeomorphism. The following universality property of the Hilbert cube was proved in [8, 11.2].

Theorem 2.3 (Approximation by Z-embedding). *For any compact metrizable space K the set of Z-embeddings is dense in the function space $C(K, \mathbb{I}^\omega)$.*

We shall apply the Z-Set Unknotting Theorem to prove the following tameness lemma.

Lemma 2.4. *For each zero-dimensional Z-set A in the Hilbert cube $\mathbb{I}^\omega$ and every open cover $\mathcal{U}$ of $\mathbb{I}^\omega$ there is a finite cover $B_1, \dots, B_n$ of A by open subsets of $\mathbb{I}^\omega$ such that*

1. *for every $i \le n$ the closure $\bar{B}_i$ is homeomorphic to $\mathbb{I}^\omega$, is contained in some set $U \in \mathcal{U}$ and the boundary ∂B_i of B_i in $\mathbb{I}^\omega$ is a Z-set in $\bar{B}_i$;*
2. *for any distinct numbers $i, j \le n$ the closures $\bar{B}_i$ and $\bar{B}_j$ are disjoint.*

Proof. This lemma is obvious for zero-dimensional sets contained in straight intervals of the form $[0,1] \times \{x_0\} \subseteq \mathbb{I} \times \mathbb{I}^\mathbb{N} = \mathbb{I}^\omega$. The general case can be reduced to this special case with help of the Z-set Unknotting Theorem 2.2. □

3. Proof of Theorem 1.4

We shall derive Theorem 1.4 from five lemmas proved below. In these lemmas by X we denote the Hilbert cube $\mathbb{I}^\omega$ and by $\mathcal{H}(X)$ its homeomorphism group endowed with the compact-open topology. A neighborhood base of this topology at each $h \in \mathcal{H}(X)$ consists of the sets

$$B(h, \mathcal{U}) = \{f \in \mathcal{H}(X) : (h, f) \prec \mathcal{U}\},$$

where $\mathcal{U}$ runs over all open covers of X. For a cover $\mathcal{U}$ of X and maps $f, g : X \to X$, we write $(f, g) \prec \mathcal{U}$ and say that f and g are *$\mathcal{U}$-near* if for every $x \in X$ the set $\{f(x), g(x)\}$ is contained in some set $U \in \mathcal{U}$.

It is well-known that for any metric d generating the topology of X, the compact-open topology on $\mathcal{H}(X)$ is generated by the complete metric

$$\hat{d}(f, g) = \sup_{x\in X} d(f(x), g(x)) + \sup_{y\in X} d(f^{-1}(y), g^{-1}(y)).$$

This implies that $\mathcal{H}(X)$ is a Polish topological group.

In the following lemmas, for subsets $F, G \subseteq X$ and a natural number $n \in \mathbb{N}$, we shall establish some properties of the subset

$$H^k_{F,G} = \big\{(h_i)_{i=1}^k \in \mathcal{H}(X)^k : F \subseteq \bigcup_{i=1}^k h_i(G)\big\}$$

of the k-th power $\mathcal{H}(X)^k$ of the homeomorphism group $\mathcal{H}(X)$.

Lemma 3.1. *For a closed set $F \subseteq X$ and an open set $U \subseteq X$ the set*

$$H^k_{F,U} = \{(h_i)_{i=1}^k \in \mathcal{H}(X)^k : F \subseteq \bigcup_{i=1}^k h_i(U)\}$$

is open in $\mathcal{H}(X)^k$ for every $k \in \mathbb{N}$.

Proof. The proof is by induction on $k \in \mathbb{N}$.

First we verify this lemma for $k = 1$. Since $\mathcal{H}(X)$ is a topological group, it suffices to check that the set

$$(H^1_{F,U})^{-1} = \{h \in \mathcal{H}(X) : F \subseteq h(U)\}^{-1} = \{h \in \mathcal{H}(X) : h^{-1}(F) \subseteq U\}^{-1} = \{h \in \mathcal{H}(X) : h(F) \subseteq U\}$$

is open in $\mathcal{H}(X)$. But this follows from the definition of the compact-open topology on $\mathcal{H}(X)$.

Now assume that the lemma has been proved for some $k \in \mathbb{N}$. To prove it for $k+1$, fix any sequence of homeomorphisms $(f_i)_{i=1}^{k+1} \in H^{k+1}_{F,U}$. Then $F \subseteq \bigcup_{i=1}^{k+1} f_i(U)$ and hence $\bigcap_{i=1}^{k+1} f_i(X \setminus U) \subseteq X \setminus F$. It follows that the closed sets $B = \bigcap_{i=1}^k f_i(X \setminus U)$ and $C = F \cap f_{k+1}(X \setminus U)$ are disjoint and hence have disjoint open neighborhoods V and $O(C)$ in X. Then V and $W = O(C) \cup (X \setminus F)$ are two open sets such that

- $V \cap W \subseteq X \setminus F$;
- $\bigcap_{i=1}^k f_i(X \setminus U) = B \subseteq V$, which is equivalent to $X \setminus V \subseteq \bigcup_{i=1}^k f_i(U)$;
- $f_{k+1}(X \setminus U) \subseteq W$, which is equivalent $X \setminus W \subseteq f_{k+1}(U)$.

By the inductive assumption, the sets $H^k_{X\setminus V,U} \ni (f_i)_{i=1}^k$ and $H^1_{X\setminus W,U} \ni f_{k+1}$ are open and so is their product

$$H^k(X \setminus V, U) \times H^1(X \setminus W, U) \subseteq \mathcal{H}(X)^k \times \mathcal{H}(X) = \mathcal{H}(X)^{k+1},$$

which contains the sequence $(f_i)_{i=1}^{k+1}$ and lies in the set $\mathcal{H}^{k+1}_{F,U}$. This shows that the set $H^{k+1}(F,U)$ is open in $\mathcal{H}(X)^{k+1}$. □

Lemma 3.2. *For an F_σ-set $F \subseteq X$ and an G_δ-set $G \subseteq X$ the set*

$$H^k_{F,G} = \big\{(h_i)_{i=1}^k \in \mathcal{H}(X)^k : F \subseteq \bigcup_{i=1}^k h_i(G)\big\}$$

is of type G_δ in $\mathcal{H}(X)^k$ for every $k \in \mathbb{N}$.

Proof. Write the F_σ-set F as the union $F = \bigcup_{j\in\omega} F_j$ of a non-decreasing sequence $(F_j)_{j\in\omega}$ of closed subsets of X and write the G_δ-set G as the intersection $G = \bigcap_{j\in\omega} U_j$ of a non-increasing sequence $(U_j)_{j\in\omega}$ of open subsets of X. Then for any sequence of homeomorphisms $(h_i)_{i=1}^k \in \mathcal{H}(X)^k$ we get

$$\bigcup_{i=1}^k h_i(G) = \bigcup_{i=1}^k \bigcap_{j\in\omega} h_i(U_j) = \bigcap_{j\in\omega} \bigcup_{i=1}^k h_i(U_j).$$

Consequently, the set

$$H^k_{F,G} = \{(h_i)_{i=1}^k \in \mathcal{H}(X)^k : \bigcup_{j\in\omega} F_j \subseteq \bigcap_{j\in\omega} \bigcup_{i=1}^k h_i(U_j)\} =$$

$$= \bigcap_{j\in\omega} \{(h_i)_{i=1}^k \in \mathcal{H}(X)^k : F_j \subseteq \bigcup_{i=0}^k h_i(U_j)\} = \bigcap_{j\in\omega} H^k_{F_j,U_j}$$

is of type G_δ in $\mathcal{H}(X)^k$ as each set $H^k_{F_j,U_j}$ is open in $\mathcal{H}(X)^k$ by Lemma 3.1. □

While the preceding two lemmas hold for any compact metrizable space X, the following three lemmas essentially depend on the properties of the Hilbert cube $X = \mathbb{I}^\omega$.

Lemma 3.3. *For any zero-dimensional Z-set $F \subseteq X$ and any dense open subset G of X the set $H^{-1}_{F,G} = \{h \in \mathcal{H}(X) : h(F) \subseteq G\}$ is a dense in $\mathcal{H}(X)$.*

Proof. Given a homeomorphism $h_0 \in \mathcal{H}(X)$ and an open cover $\mathcal{U}$ of the Hilbert cube $X = \mathbb{I}^\omega$, we need to find a homeomorphism h of $\mathbb{I}^\omega$ such that $h(F) \subseteq G$ and $(h, h_0) \prec \mathcal{U}$.

By Lemma 2.4, there is a finite cover $B_1, \dots, B_n$ of the zero-dimensional Z-set $h_0(F)$ by open subsets of X such that

- for every $i \le n$ the closure $\bar{B}_i$ of B_i lies in some set $U \in \mathcal{U}$, $\bar{B}_i$ is homeomorphic to the Hilbert cube and ∂B_i is a Z-set in $\bar{B}_i$;
- for any $1 \le i \ne j \le n$ the closures $\bar{B}_i$ and $\bar{B}_j$ are disjoint.

For every $i \le n$ we shall construct a homeomorphism $f_i : \bar{B}_i \to \bar{B}_i$ such that $f_i|\partial B_i = \mathrm{id}$ and $f_i(B_i \cap F_0) \subseteq G$. Using the Approximation Theorem 2.3, in the function space $C(\mathbb{I}^\omega, \bar{B}_i)$ choose a countable dense subset $\{g_j\}_{j\in\omega}$ consisting of Z-embeddings. Then $A_i = \bigcup_{j\in\omega} g_j(\mathbb{I}^\omega)$ is a σZ-set in $\bar{B}_i$ and each closed subset $C \subseteq X \setminus A_i$ of $\bar{B}_i$ is a Z-set in $\bar{B}_i$. Taking into account that G is an open dense subset of X, we conclude that the set $B_i \cap G \setminus A$ is non-meager in X. Consequently, this set is uncountable, which allows us to find a topological copy of the Cantor cube in $B_i \cap U \setminus A$. Since the Cantor cube contains a topological copy of each zero-dimensional compact metrizable space, we can find a compact subset $K_i \subseteq B_i \cap U \setminus A_i$, homeomorphic to the compact zero-dimensional set $\bar{B}_i \cap h_0(F) = B_i \cap h_0(F)$. It follows from $K_i \cap A_i = \emptyset$ that K_i is a Z-set in $\bar{B}_i$. Using the fact that $h_0(F)$ is a Z-set in $\mathbb{I}^\omega$ and ∂B_i is a Z-set in $\bar{B}_i$, we can show that $h_0(F) \cap B_i$ is a Z-set in $\bar{B}_i$. By the Z-set Unknotting Theorem, there is a homeomorphism $f_i : \bar{B}_i \to \bar{B}_i$ such that $f_i|\partial B_i = \mathrm{id}$ and $f_i(h_0(F) \cap B_i) = K_i \subseteq U$. The homeomorphisms f_i, $1 \le i \le n$, compose a homeomorphism $f : X \to X$ such that $f|\bar{B}_i = f_i$, $i \le n$ and $f|X \setminus \bigcup_{i=1}^n B_i = \mathrm{id}$. The homeomorphism f is $\mathcal{U}$-near to the identity homeomorphism of X and $f(h_0(F)) \subseteq U$. Then the homeomorphism $h = f \circ h_0 : X \to X$ is $\mathcal{U}$-near to h_0 and $h(F) \subseteq U$. □

Lemma 3.4. *For every dense G_δ-set G in $\mathbb{I}^\omega$ and every σZ-set $F \subseteq \mathbb{I}^\omega$ of finite dimension $k = \dim(F)$ the set*

$$H^{k+1}_{F,G} = \left\{(h_i)_{i=1}^{k+1} \in \mathcal{H}(X)^{k+1} : F \subseteq \bigcup_{i=1}^{k+1} h_i(G)\right\}$$

is dense G_δ in the space $\mathcal{H}(X)^{k+1}$.

Proof. By Lemma 3.2, the set $H^{k+1}_{F,G}$ is of type G_δ in $\mathcal{H}(X)^{k+1}$. So, it remains to prove that this set is dense in $\mathcal{H}(X)^{k+1}$. This will be done by induction on $k \in \mathbb{N}$.

First we check the lemma for $k = 0$. Fix a σZ-subset F in X of dimension $\dim(F) = 0$ and consider the subset $H^1_{F,G} = \{h \in \mathcal{H}(X) : A \subseteq h(G)\}$ of $\mathcal{H}(X)$. Write the σZ-set F as the union $F = \bigcup_{j\in\omega} F_j$ of an increasing sequence $(F_j)_{j\in\omega}$ of Z-sets in X and write the dense G_δ-set G as the intersection $G = \bigcap_{j\in\omega} U_j$ of a decreasing sequence $(U_j)_{j\in\omega}$ of open dense subsets of X. Observe that

$$H^1_{F,G} = \bigcap_{j\in\omega} H^1_{F_j,U_j}.$$

By Lemma 3.3, for every $j \in \omega$ the set

$$(H^1_{F_j,U_j})^{-1} = \{h \in \mathcal{H}(X) : F_j \subseteq h(U_j)\}^{-1} = \{h \in \mathcal{H}(X) : h(F_j) \subseteq U_i\}$$

is dense in $\mathcal{H}(X)$ and so is its inverse $H^1_{F_j,U_j}$. By Lemma 3.1, the set $H^1_{F_j,U_j}$ is open in $\mathcal{H}(X)$. Then the set $H^1_{F,G} = \bigcap_{i\in\omega} H^1_{F_j,U_j}$ is a dense G_δ in $\mathcal{H}(X)$, being a countable intersection of open dense sets in the Polish space $\mathcal{H}(X)$.

Now assume that the lemma has been proven for some $k \in \omega$. Given any σZ-set $F \subseteq \mathbb{I}^\omega$ of dimension $\dim(F) = k+1$, we need to prove that the set $H^{k+2}_{F,G}$ is dense in $\mathcal{H}(X)^{k+2}$. Fix any non-empty open set $\mathcal{U} \subseteq \mathcal{H}(X)^{k+2} = \mathcal{H}(X)^{k+1} \times \mathcal{H}(X)$. We can assume that $\mathcal{U}$ is of the form $\mathcal{U} = \mathcal{V} \times \mathcal{W}$ for some open sets $\mathcal{V} \subseteq \mathcal{H}(X)^{k+1}$ and $\mathcal{W} \subseteq \mathcal{H}(X)$.

The space F has (inductive) dimension $k+1$ and hence F has a countable base $\mathcal{B} = \{U_j : j \in \omega\}$ of the topology such that the boundary ∂U_j of each set $U_j \in \mathcal{B}$ in the space F has dimension $\dim(\partial U_j) \le k$. By the inductive assumption, for every $j\in\omega$ the set

$$H^{k+1}_{\partial U_j,G}=\big\{(h_i)_{i=1}^{k+1}\in\mathcal H(X)^{k+1}:\partial U_j\subseteq\bigcup_{i=1}^{k+1}h_i(G)\big\}$$

is dense G_δ in $\mathcal H(X)^{k+1}$. Then the intersection $\bigcap_{j\in\omega}H^{k+1}_{\partial U_j,G}$ also is a dense G_δ-set in $\mathcal H(X)^{k+1}$. So, we can choose a sequence of homeomorphisms $(h_i)_{i=1}^{k+1}\in\mathcal V\cap\bigcap_{j\in\omega}H^{k+1}_{\partial U_j,G}$. For these homeomorphisms, we get $\bigcup_{j\in\omega}\partial U_j\subseteq\bigcup_{i=1}^{k+1}h_i(G)$.

Now consider the σZ-set $F'=F\setminus\bigcup_{i=1}^{k+1}h_i(G)$. Since $\{U_j\}_{j\in\omega}$ is the base of the topology of the space F and the intersection $F'\cap\bigcup_{j\in\omega}\partial U_j$ is empty, the set F' has dimension zero. By the inductive assumption (for $k=0$) the set $H^1_{F',G}$ is dense G_δ in $\mathcal H(X)$, which allows us to find a homeomorphism $h_{k+2}\in\mathcal W\cap H^1_{F',G}$. Then the sequence of homeomorphisms $(h_i)_{i=0}^{k+2}$ belongs to the set $(\mathcal V\times\mathcal W)\cap H^{k+2}_{F,G}$ witnessing that the set $H^{k+2}_{F,U}$ is dense in $\mathcal H(X)^{k+2}$. □

Lemma 3.5. *For every dense G_δ-set G in $X=\mathbb I^\omega$ and every F_σ-set $F\subseteq\mathbb I^\omega$ of finite dimension $k=\dim(F)$ the set*

$$H^{k+4}_{F,G}=\big\{(h_i)_{i=1}^{k+4}\in\mathcal H(X)^{k+1}:F\subseteq\bigcup_{i=1}^{k+4}h_i(G)\big\}$$

is dense G_δ in the space $\mathcal H(X)^{k+4}$.

Proof. Given any non-empty open set $\mathcal U\subseteq\mathcal H(X)^{k+4}$, we need to show that $\mathcal U\cap H^{k+4}_{F,G}\ne\emptyset$. We lose no generality assuming that $\mathcal U=\mathcal V\times\mathcal W$ for some non-empty open sets $\mathcal V\in\mathcal H^{k+1}(X)$ and $\mathcal W\subseteq\mathcal H^3(X)$.

By Theorem 2.3, the function space $C(\mathbb I^2,X)$ contains a dense countable subset $\{f_j\}_{j\in\omega}$ consisting of Z-embeddings. It follows that $A=\bigcup_{j\in\omega}f_j(\mathbb I^2)$ is a σZ-set of dimension $\dim(A)=2$ in $\mathbb I^\omega$. By Lemma 3.4, the set $H^3_{A,G}$ is dense in $\mathcal H^3(G)$. Consequently, we can find homeomorphisms $(h_{k+2},h_{k+3},h_{k+4})\in\mathcal W$ such that $A\subseteq\bigcup_{i=k+2}^{k+4}h_i(G)$. Now consider the G_δ-set $G'=\bigcup_{i=k+2}^{k+4}h_i(G)$ and the finite-dimensional F_σ-set $F\setminus G'\subseteq X\setminus A$ in X. It follows from the choice of the set A that $F\setminus G'$ is a σZ_2-set. Since each finite-dimensional Z_2-set in the Hilbert cube is a Z-set (see [16]), the finite-dimensional σZ_2-set $F\setminus G'$ is a σZ-set in $X=\mathbb I^\omega$. By Lemma 3.4, the set $H^{k+1}_{F\setminus G',G}$ is dense in $\mathcal H(X)^{k+1}$, which allows us to find homeomorphisms $(h_1,\dots,h_{k+1})\in\mathcal V$ such that $F\setminus G'\subseteq\bigcup_{i=1}^{k+1}h_i(G)$. Then $(h_1,\dots,h_{k+1},h_{k+2},h_{k+3},h_{k+4})\in\mathcal V\times\mathcal W=\mathcal U$ is a sequence of homeomorphisms with $F\subseteq\bigcup_{i=1}^{k+4}h_i(G)$, which means that this sequence belongs to the set $H^{k+4}_{F,G}$. □

Proof of Theorem 1.4. Let $\mathcal I\not\subseteq\mathcal M$ be any topologically invariant ideal with BP-base on the Hilbert cube $\mathbb I^\omega$. Repeating the argument of the proof of Theorem 1.1, we can show that $\mathcal I$ contains a dense G_δ-set G in $\mathbb I^\omega$. By Lemma 3.5, for any closed subset $F\subseteq\mathbb I^\omega$ of finite dimension $k=\dim(F)$ there are homeomorphisms $h_1,\dots,h_{k+4}\in\mathcal H(\mathbb I^\omega)$ such that $F\subseteq\bigcup_{i=1}^{k+4}h_i(G)$. By the topological invariantness and additivity of $\mathcal I$, the union $\bigcup_{i=1}^{k+4}h_i(G)$ and its subset F belong to the ideal $\mathcal I$. So, $\overline{\mathcal D}_\omega\subseteq\mathcal I$.

If $\mathcal I$ is a σ-ideal, the $\sigma\overline{\mathcal D}_\omega\subseteq\mathcal I$. □

4. Proof of Theorem 1.5

The proof of Theorem 1.5, is divided into four lemmas: 4.4, 4.5, 4.6, and 4.7 reducing the problem of calculation of the cardinal characteristics of the ideal $\sigma\mathcal C_0$ to zero-dimensional level. The reduction will be made with help of semi-open bijection of the Baire space $\mathbb Z^\omega$ onto the Hilbert cube.

A map $f:X\to Y$ between topological spaces is called *semi-open* if for each non-empty open set $U\subseteq X$ the image $f(U)$ has non-empty interior in Y. The following property of bijective semi-open maps is immediate.

Lemma 4.1. *If $f:X\to Y$ is a bijective semi-open map between topological spaces, then for any nowhere dense subset $A\subseteq X$ its image $f(A)$ is nowhere dense in Y.*

A standard example of a semi-open map is the Cantor ladder map

$$c:\{0,1\}^\omega\to[0,1],\quad c:(x_i)_{i=1}^\infty\mapsto\sum_{i=1}^\infty\frac{x_i}{2^{i+1}}$$

of the Cantor cube $\{0,1\}^\omega$ onto the closed interval $[0,1]$. This map will be used to prove:

Lemma 4.2. *There exists a bijective semi-open map $\varphi : \mathbb{Z}^\omega \to \mathbb{I}$ of the Baire space $\mathbb{Z}^\omega$ onto the closed interval $\mathbb{I}$.*

Proof. Consider the Cantor ladder map $c : \{0,1\}^\omega \to [0,1]$. It is well-known that for each point y of the set $Q_2 = \{\frac{m}{2^k} : 0 < m < 2^k,\ k, m \in \omega\} \subseteq [0,1]$ the preimage $c^{-1}(y)$ has cardinality $|c^{-1}(y)| = 2$ and for every $y \in [0,1] \setminus Q_2$ the preimage $c^{-1}(y)$ is a singleton. Take any subset $B \subseteq \{0,1\}^\omega$ such that the restriction $c|B : B \to [0,1]$ is bijective. It follows that the set B is dense and has countable complement in $\{0,1\}^\omega$. Consequently, B is zero-dimensional Polish nowhere locally compact space, which is homeomorphic to the Baire space $\mathbb{Z}^\omega$ according to the Aleksandrov-Urysohn Theorem [15, 7.7]. It is easy to see that the restriction $c|B : B \to [0,1]$ is semi-open. Then for any homeomorphism $h : \mathbb{Z}^\omega \to B$ the map $\varphi = c \circ h : \mathbb{Z}^\omega \to [0,1]$ is a bijective semi-open map of the Baire space $\mathbb{Z}^\omega$ onto the interval $[0,1]$. □

We shall consider the Baire space $\mathbb{Z}^\omega$ as a topological group endowed with the operation of addition of functions. In this group consider the closed nowhere dense subset $\mathbb{Z}_0^\omega$ where $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$.

Lemma 4.3. *There is a bijective semi-open map $\Phi : \mathbb{Z}^\omega \to \mathbb{I}^\omega$ such that for every $f \in \mathbb{Z}^\omega$ the set $\Phi(f + \mathbb{Z}_0^\omega)$ belongs to the σ-ideal $\sigma\mathcal{C}_0$ generated by zero-dimensional Z-set in $\mathbb{I}^\omega$.*

Proof. Take the bijective semi-open map $\varphi : \mathbb{Z}^\omega \to \mathbb{I}$ from Lemma 4.2 and consider its countable power

$$\varphi^\omega : (\mathbb{Z}^\omega)^\omega \to \mathbb{I}^\omega, \ \ \varphi^\omega : (x_i)_{i\in\omega} \mapsto (\varphi(x_i))_{i\in\omega}.$$

For each function $f \in (\mathbb{Z}^\omega)^\omega$ the set $f + (\mathbb{Z}_0^\omega)^\omega$ can be written as the countable product $\prod_{n\in\omega}(f_n + \mathbb{Z}_0^\omega)$ for suitable functions $f_n \in \mathbb{Z}^\omega$, $n \in \omega$. Then $\varphi^\omega(f + (\mathbb{Z}_0^\omega)^\omega) = \prod_{n\in\omega} \varphi(f_n + \mathbb{Z}_0^\omega)$. Observe that for every $n \in \omega$ the set $f_n + \mathbb{Z}_0^\omega$ is nowhere dense in $\mathbb{Z}^\omega$. Since the map φ^ω is bijective and semi-open, the image $\varphi(f_n + \mathbb{Z}_0^\omega)$ is nowhere dense in the interval $\mathbb{I}$ and so is its closure K_n in $\mathbb{I}$. By Lemma 2.1, the product $K = \prod_{n\in\omega} K_n$ is a zero-dimensional Z-set in $\mathbb{I}^\omega$. Consequently, the set $\varphi^\omega(f + (\mathbb{Z}_0^\omega)^\omega) \subseteq K$ belongs to the ideal $\sigma\mathcal{C}_0$. Then for any coordinate permutating homeomorphism $h : \mathbb{Z}^\omega \to (\mathbb{Z}^\omega)^\omega$ the map $\Phi = \varphi^\omega \circ h : \mathbb{Z}^\omega \to \mathbb{I}^\omega$ has the required property: $\Phi(f + \mathbb{Z}_0^\omega) \in \sigma\mathcal{C}_0$ for every $f \in \mathbb{Z}^\omega$. □

Lemma 4.4. $\mathrm{cov}(\sigma\mathcal{C}_0) = \mathrm{cov}(\mathcal{M})$.

Proof. The inequality $\mathrm{cov}(\sigma\mathcal{C}_0) \geq \mathrm{cov}(\mathcal{M})$ is obvious, because $\sigma\mathcal{C}_0 \subseteq \mathcal{M}$. The proof of the inequality $\mathrm{cov}(\sigma\mathcal{C}_0) \leq \mathrm{cov}(\mathcal{M})$ uses the equality

$$\mathrm{cov}(\mathcal{M}) = \min\{|\mathcal{F}| : \ \mathcal{F} \subseteq \mathbb{Z}^\omega \text{ and } \mathcal{F} + \mathbb{Z}_0^\omega = \mathbb{Z}^\omega\}$$

proved in Theorem 2.4.1 [5]. According to this equality, there is a subset $\mathcal{F} \subseteq \mathbb{Z}^\omega$ of cardinality $|\mathcal{F}| = \mathrm{cov}(\mathcal{M})$ such that $\mathbb{Z}_0^\omega + \mathcal{F} = \mathbb{Z}^\omega$.

By Lemma 4.3, there is a bijective map $\Phi : \mathbb{Z}^\omega \to \mathbb{I}^\omega$ such that for every $f \in \mathbb{Z}^\omega$ the set $\Phi(f + \mathbb{Z}_0^\omega)$ belongs to the ideal $\sigma\mathcal{C}_0$. Since

$$\mathbb{I}^\omega = \Phi(\mathbb{Z}^\omega) = \bigcup_{f\in\mathcal{F}} \Phi(f + \mathbb{Z}_0^\omega),$$

the family $\{\Phi(f + \mathbb{Z}_0^\omega)\}_{f\in\mathcal{F}} \subseteq \sigma\mathcal{C}_0$ is a cover of $\mathbb{I}^\omega$, witnessing that $\mathrm{cov}(\sigma\mathcal{C}_0) \leq |\mathcal{F}| = \mathrm{cov}(\mathcal{M})$. □

Lemma 4.5. $\mathrm{non}(\sigma\mathcal{C}_0) = \mathrm{non}(\mathcal{M})$.

Proof. The inequality $\mathrm{non}(\sigma\mathcal{C}_0) \leq \mathrm{non}(\mathcal{M})$ is obvious, since $\sigma\mathcal{C}_0 \subseteq \mathcal{M}$. To prove the inequality $\mathrm{non}(\sigma\mathcal{C}_0) \geq \mathrm{non}(\mathcal{M})$, we shall use a combinatorial characterization of the cardinal $\mathrm{non}(\mathcal{M})$ due to Bartoszynski [5, 2.4.7]. According to this characterization, $\mathrm{non}(\mathcal{M})$ coincides with the smallest cardinality of a subset $A \subseteq \mathbb{Z}^\omega$ which cannot be covered by countably many sets of the form $f + \mathbb{Z}_0^\omega$, $f \in \mathbb{Z}^\omega$.

Let $\Phi : \mathbb{Z}^\omega \to \mathbb{I}^\omega$ be the bijective map from Lemma 4.3. Observe that for any subset $A \subseteq \mathbb{I}^\omega$ of cardinality $|A| < \mathrm{non}(\mathcal{M})$ its preimage $\Phi^{-1}(A) \subseteq \mathbb{Z}^\omega$ has cardinality $|\Phi^{-1}(A)| = |A| < \mathrm{non}(\mathcal{M})$ and by the combinatorial characterization of $\mathrm{non}(\mathcal{M})$, can be covered by the set $C + \mathbb{Z}_0^\omega$ for some countable set $C \subseteq \mathbb{Z}^\omega$. Then $A \subseteq \bigcup_{f\in C} \Phi(f + \mathbb{Z}_0^\omega) \in \sigma\mathcal{C}_0$. This implies that $\mathrm{non}(\sigma\mathcal{C}_0) \geq \mathrm{non}(\mathcal{M})$. □

Lemma 4.6. $\mathrm{add}(\sigma\mathcal{C}_0, \mathcal{M}) = \mathrm{add}(\mathcal{M})$.

Proof. The inequality $\mathrm{add}(\sigma\mathcal{C}_0,\mathcal{M})\ge\mathrm{add}(\mathcal{M})$ is trivial. Since $\mathrm{add}(\mathcal{M})=\min\{\mathrm{cov}(\mathcal{M}),\mathfrak{b}\}$, the inequality $\mathrm{add}(\sigma\mathcal{C}_0,\mathcal{M})\le\mathrm{add}(\mathcal{M})$ will follow as soon as we check that $\mathrm{add}(\sigma\mathcal{C}_0,\mathcal{M})\le\min\{\mathrm{cov}(\mathcal{M}),\mathfrak{b}\}$. Lemma 4.4 implies that $\mathrm{add}(\sigma\mathcal{C}_0,\mathcal{M})\le\mathrm{cov}(\sigma\mathcal{C}_0)=\mathrm{cov}(\mathcal{M})$.

To prove that $\mathrm{add}(\sigma\mathcal{C}_0,\mathcal{M})\le\mathfrak{b}$, consider the set $\mathbb{I}\setminus\mathbb{Q}$ of irrational numbers in $\mathbb{I}$ and its countable power $(\mathbb{I}\setminus\mathbb{Q})^\omega$, which is homeomorphic to the Baire space $\mathbb{Z}^\omega$ according to the Aleksandrov-Urysohn Theorem [15, 7.7]. By Theorem 2.2.3 of [5], the space $(\mathbb{I}\setminus\mathbb{Q})^\omega$ (being a topological copy of $\mathbb{Z}^\omega$) contains a family $\mathcal{A}$ of compact subsets of cardinality $|\mathcal{A}|=\mathfrak{b}$ whose union $\bigcup\mathcal{A}$ is non-meager in $(\mathbb{I}\setminus\mathbb{Q})^\omega$ and hence is non-meager in the Hilbert cube $\mathbb{I}^\omega$. By Lemma 2.1, each set $A\in\mathcal{A}$ is a zero-dimensional Z-set in $\mathbb{I}^\omega$ and hence $\mathcal{A}\subseteq\sigma\mathcal{C}_0$. Since $\bigcup\mathcal{A}\notin\mathcal{M}$, we see that $\mathrm{add}(\sigma\mathcal{C}_0,\mathcal{M})\le|\mathcal{A}|=\mathfrak{b}$. □

Lemma 4.7. $\mathrm{cof}(\sigma\mathcal{C}_0,\mathcal{M})=\mathrm{cof}(\mathcal{M})$.

Proof. The inequality $\mathrm{cof}(\sigma\mathcal{C}_0,\mathcal{M})\le\mathrm{cof}(\mathcal{M})$ is trivial. Since $\mathrm{cof}(\mathcal{M})=\max\{\mathrm{non}(\mathcal{M}),\mathfrak{d}\}$, the inequality $\mathrm{cof}(\sigma\mathcal{C}_0,\mathcal{M})\ge\mathrm{cof}(\mathcal{M})$ will follow as soon as we check that $\mathrm{cof}(\sigma\mathcal{C}_0,\mathcal{M})\ge\max\{\mathrm{non}(\mathcal{M}),\mathfrak{d}\}$. Lemma 4.5 implies that $\mathrm{cof}(\sigma\mathcal{C}_0,\mathcal{M})\ge\mathrm{non}(\sigma\mathcal{C}_0)=\mathrm{non}(\mathcal{M})$.

To prove that $\mathrm{cof}(\sigma\mathcal{C}_0,\mathcal{M})\ge\mathfrak{d}$, consider the set $\mathbb{I}\setminus\mathbb{Q}$ of irrational numbers in $\mathbb{I}$ and its countable power $P=(\mathbb{I}\setminus\mathbb{Q})^\omega$, which is homeomorphic to the Baire space $\mathbb{Z}^\omega$ according to the Aleksandrov-Urysohn Theorem [15, 7.7]. Let $\mathcal{M}(P)$ be the ideal of meager sets in P and $\sigma\mathcal{K}(P)$ be the σ-ideal generated by compact subsets of P. By Theorem 2.2.3 of [5], $\mathrm{cof}(\sigma\mathcal{K}(P),\mathcal{M}(P))=\mathfrak{d}$. Taking into account that $\sigma\mathcal{K}(P)\subseteq\sigma\mathcal{C}_0$ (which follows from Lemma 2.1) and $\mathcal{M}(P)=\{M\cap P:M\in\mathcal{M}\}$, we see that

$$\mathfrak{d}=\mathrm{cof}(\sigma\mathcal{K}(P),\mathcal{M}(P))\le\mathrm{cof}(\sigma\mathcal{C}_0,\mathcal{M}).$$

Therefore, $\mathrm{cof}(\sigma\mathcal{C}_0,\mathcal{M})\ge\max\{\mathrm{non}(\mathcal{M}),\mathfrak{d}\}=\mathrm{cof}(\mathcal{M})$ and we are done. □

5. Proof of Theorem 1.6

First we elaborate some tools for working with tame G_δ-sets in the Hilbert cube $\mathbb{I}^\omega$. We shall need an index-free description of tame G_δ-sets, developed in [4].

A family $\mathcal{T}$ of open subsets of a topological space X is defined to be *tame* if

- $\mathcal{T}$ is *vanishing* in the sense that for each open cover $\mathcal{U}$ of X the family $\{B\in\mathcal{T}:\forall U\in\mathcal{U}\ \ B\not\subseteq U\}$ is locally finite;
- for any distinct sets $A,B\in\mathcal{T}$ one of three possibilities holds: $\bar A\cap\bar B=\emptyset$, $\bar A\subseteq B$, or $\bar B\subseteq A$.

For a family $\mathcal{T}$ of subsets of a set X consider the set

$$\bigcup\nolimits^\infty\mathcal{T}=\bigcap\big\{\bigcup(\mathcal{T}\setminus\mathcal{F}):\mathcal{F}\text{ is a finite subfamily of }\mathcal{T}\big\}$$

of all points $x\in X$ that belong to infinitely many sets of the family $\mathcal{T}$.

The following characterization of tame G_δ-sets was proved in Proposition 2 of [4].

Proposition 5.1. *A subset $G\subseteq\mathbb{I}^\omega$ is a tame G_δ-set in $\mathbb{I}^\omega$ if and only if $G=\bigcup^\infty\mathcal{T}$ for a tame family $\mathcal{T}$ of tame open balls in $\mathbb{I}^\omega$.*

A subset $G\subseteq\mathbb{I}$ will be called a *tame G_δ-set* in $\mathbb{I}$ if for any non-empty open set $U\subseteq\mathbb{I}$ the complement $U\setminus G$ is uncountable.

To establish some structural properties of tame G_δ-sets in $\mathbb{I}$, we need indexed modifications of the notions of vanishing and disjoint families. An indexed family $(X_\alpha)_{\alpha\in A}$ of subsets of a compact metrizable space X is called

- *disjoint* if $X_\alpha\cap X_\beta=\emptyset$ for any distinct indexes $\alpha,\beta\in A$;
- *vanishing* if for each open cover $\mathcal{U}$ of X there set $\{\alpha\in A:\forall U\in\mathcal{U}\ \ X_\alpha\not\subseteq U\}$ is finite.

Lemma 5.2. *If $G\subseteq\mathbb{I}$ is a dense tame G_δ-set in $\mathbb{I}$, then for any non-empty open connected subset $U\subsetneq\mathbb{I}$ and any $\varepsilon>0$ there is a sequence $(U_m)_{m\in\omega}$ of non-empty open connected subsets of $\mathbb{I}$ such that*

(1) *the indexed family $(\bar U_m)_{m\in\omega}$ is disjoint;*
(2) $\bigcup_{m\in\omega}\bar U_m\subseteq U$;
(3) $\mathrm{diam}(U_m)<\varepsilon$ *for all $m\in\omega$;*
(4) $V\cap G\subseteq\bigcup_{m\in\omega}U_m$.

Proof. Being a proper open connected subset of $\mathbb{I}$, the set U is equal to (a,b), $[0,b)$, or $(b,1]$ for some numbers $0 \le a < b \le 1$. So, we can choose a disjoint sequence $(V_m)_{m\in\omega}$ of non-empty open connected subsets $\mathbb{I}$ such that

(a) $\bigcup_{m\in\omega} V_m$ is dense in U;
(b) $\bigcup_{m\in\omega} \bar{V}_m \subseteq U$;
(c) the family $\{V_m\}_{m\in\omega}$ is locally finite in U;
(d) $\mathrm{diam} V_m < \varepsilon/2$ for all $m \in \omega$.

Since the G_δ-set G is tame, for every $m \in \omega$ the complement $V_m \setminus G$ is uncountable, and hence contains a topological copy K_m of the Cantor cube K_m. The condition (c) implies that the union $K = \bigcup_{m\in\omega} K_m$ is a closed subset without isolated points in V. Consequently, its complement $V \setminus K$ can be written as the countable union $\bigcup_{m\in\omega} U_m$ of a disjoint sequence $(U_m)_{m\in\omega}$ of open connected subsets of $\mathbb{I}$ such that the family $(\bar{U}_m)_{m\in\omega}$ is disjoint. The condition (d) guarantees that $\mathrm{diam}(U_m) < \varepsilon$ for all $m \in \omega$. The obvious inclusion $V \cap G \subseteq V \setminus K = \bigcup_{m\in\omega} U_m$ completes the proof of the lemma. □

Using Lemma 5.2, by a standard inductive argument, one can prove:

Lemma 5.3. *If a dense G_δ-subset G of $\mathbb{I}$ is tame, then $G = \bigcap_{n\in\omega}\bigcup_{m\in\omega} U_{n,m}$ for some vanishing indexed family $(U_{n,m})_{n,m\in\omega}$ of open connected subsets of $\mathbb{I}$ such that for every $n \in \mathbb{N}$ the indexed family $(\bar{U}_{n,m})_{m\in\omega}$ is disjoint and the family $\{\bar{U}_{n+1,m}\}_{m\in\omega}$ refines the family $\{U_{n,m}\}_{m\in\omega}$.*

Lemma 5.4. *For each uncountable cardinal $\kappa \le \mathfrak{c}$ there is a family $(G_\alpha)_{\alpha\in\kappa}$ of dense tame G_δ-sets in $\mathbb{I}$ such that each subset $X \subseteq \mathbb{I}$ of cardinality $|X| < \kappa$ is contained in some set G_α, $\alpha \in \kappa$.*

Proof. Fix a countable base $(U_n)_{n\in\omega}$ of the topology of the interval $\mathbb{I}$ and in each set U_n fix a disjoint family $(C_{n,\alpha})_{\alpha\in\kappa}$ of κ many Cantor sets. Observe that for every $\alpha \in \kappa$ the complement $G_\alpha = \mathbb{I} \setminus \bigcup_{n\in\omega} C_{n,\alpha}$ is a dense tame G_δ-set in $\mathbb{I}$.

Given any subset $X \subseteq \mathbb{I}$ of cardinality $|X| < \kappa$, for every $n \in \omega$ consider the set $A_n = \{\alpha \in \kappa : X \cap C_{n,\alpha} \ne \emptyset\}$ and observe that it has cardinality $|A_n| \le |X| < \kappa$. Then the union $A = \bigcup_{n\in\omega} A_n$ also has cardinality $|A| < \kappa$ and we can choose an ordinal $\alpha \in \kappa \setminus A$. For this ordinal α we get $X \subseteq \mathbb{I} \setminus \bigcup_{n\in\omega} C_{n,\alpha} = G_\alpha$. □

Now we are ready to prove the principal ingredient of the proof of Theorem 1.6.

Proposition 5.5. *Let G be a dense tame G_δ-set in the unit interval $\mathbb{I}$. Then:*

(1) *the countable power G^ω can be covered by $\mathfrak{b}$ many tame G_δ-sets in $\mathbb{I}^\omega$;*
(2) *any subset $X \subseteq G^\omega$ of cardinality $|X| < \mathfrak{d}$ can be covered by a single tame G_δ-set in $\mathbb{I}^\omega$.*

Proof. By Lemma 5.3, $G = \bigcap_{n\in\omega}\bigcup_{m\in\omega} U_{n,m}$ for some vanishing indexed family $\mathcal{U} = (U_{n,m})_{n,m\in\omega}$ of open connected subsets of $\mathbb{I}$ such that for every $n \in \mathbb{N}$ the indexed family $(\bar{U}_{n,m})_{m\in\omega}$ is disjoint and the family $\{\bar{U}_{n+1,m}\}_{m\in\omega}$ refines the family $\{U_n, m\}_{m\in\omega}$.

For every increasing function $f : \omega \to \omega$ we define a tame G_δ-set $\mathcal{U}^f$ in $\mathbb{I}^\omega$ as follows. For every $n \in \omega$ let

$$a_n^f = \max\{\sup U_{n,m} : m \le f(n),\ 1 \notin \bar{U}_{n,m}\} \text{ and}$$
$$b_n^f = \min(\{1\} \cup \{\inf U_{n,m} : m \le f(n),\ 1 \in \bar{U}_{n,m}\}).$$

Since the indexed family $(\bar{U}_{n,m})_{m\in\omega}$ is disjoint, $0 < a_n^f < b_n^f \le 1$. Moreover, $\bigcup_{m\le f(n)} U_{n,m} \subseteq [0, a_n^f) \cup (b_n^f, 1]$.

Now for every $n \in \omega$ consider the finite family

$$\mathcal{T}_n^f = \Big\{ \prod_{i\in\omega} V_i : V_0, \dots, V_{n-1} \in \{U_{n,m}\}_{m\le f(n)},\ V_n \in \{[0,a_n^f), (b_n^f, 1]\},\ V_i = \mathbb{I} \text{ for all } i > n \Big\}$$

of tame open balls in the Hilbert cube $\mathbb{I}^\omega$.

Claim 5.6. *The family $\{\bar{V} : V \in \mathcal{T}_n^f\}$ is disjoint.*

Proof. This follows immediately from the fact that the family $\{\bar{U}_{n,m}\}_{m\in\omega}$ is disjoint. □

Claim 5.7. *The family $\mathcal{T}^f = \bigcup_{n\in\omega} \mathcal{T}_n^f$ is tame.*

Proof. We need to check two conditions from the definition of a tame family.

1. The vanishing property of $\mathcal{T}$ will follow as soon as we check that for each $\varepsilon > 0$ the subfamily $\{V \in \mathcal{T} : \operatorname{diam}(V) \geq \varepsilon\}$ is finite. Here we consider the metric

$$d\big((x_n)_{n\in\omega}, (y_n)_{n\in\omega}\big) = \max_{n\in\omega} \frac{|x_n - y_n|}{2^n}$$

on the Hilbert cube $\mathbb{I}^\omega$.

Since the double sequence $(U_{n,m})_{m\in\omega}$ is vanishing, there is $n \in \omega$ so large that $2^{-n} < \varepsilon$ and $\operatorname{diam}(U_{k,m}) < \varepsilon$ for all $k \geq n$ and all $m \in \omega$. Then for every $k \geq n$, every tame open ball $V \in \mathcal{T}^f_k$ has diameter $\operatorname{diam}(V) < \varepsilon$. This implies that the family $\{V \in \mathcal{T}^f : \operatorname{diam}(V) \geq \varepsilon\} \subseteq \bigcup_{k<n} \mathcal{T}^f_k$ is finite.

2. Given two distinct sets $V, W \in \mathcal{T}^f$, we need to check that $\overline{V} \cap \overline{W} = \emptyset$, $\overline{V} \subseteq W$ or $\overline{W} \subseteq V$. Find numbers $k, n \in \omega$ such that $V \in \mathcal{T}^f_k$ and $W \in \mathcal{T}^f_n$. We lose no generality assuming that $k \leq n$. If $n = k$, then $\bar{V} \cap \bar{W} = \emptyset$ by Claim 5.6. Next, consider the case $k < n$. It follows that $V = \prod_{i\in\omega} V_i$ and $W = \prod_{i\in\omega} W_i$ for some sets $V_0, \dots, V_{k-1} \in \{U_{k,m}\}_{m\leq f(k)}$, $V_k \in \{[0, a^f_k), (b^f_k, 1]\}$, $V_i = \mathbb{I}$ for $i > k$, and $W_0, \dots, W_{n-1} \in \{U_{n,m}\}_{m\leq f(n)}$, $W_n \in \{[0, a^f_n), (b^f_n, 1]\}$, and $W_i = \mathbb{I}$ for $i > n$. The choice of the family $\{U_{i,j}\}_{i,j\in\omega}$ guarantees that either $\bar{V}_i \cap \bar{W}_i = \emptyset$ for some $i < k$ or else $\bar{W}_i \subseteq V_i$ for all $i < k$. In the first case the sets V and W have disjoint closures. So, it remains to consider the second case: $\bar{W}_i \subseteq V_i$ for all $i < k$. Consider the set W_k, which is equal to $U_{n,m}$ for some $m \leq f(n)$. It follows that $\bar{U}_{n,m} \subseteq U_{k,m'}$ for some number $m' \in \omega$. Since the family $\{\bar{U}_{k,i}\}_{i\in\omega}$ is disjoint and consists of connected subsets of $\mathbb{I}$ four cases are possible: (i) $U_{k,m'} \subseteq [0, a^f_k)$, (ii) $b^f_k < 1$ and $U_{k,m'} \subseteq (b^f_k, 1]$, (iii) $b^f_k < 1$ and $\bar{U}_{k,m'} \subseteq (a^f, b^f)$, and (iv) $b^f_k = 1$ and $\bar{U}_{k,m'} \subseteq (a^f_k, 1]$. In the case (i) we get $\overline{W} \subseteq V$ if $V_k = [0, a^f_k)$ and $\overline{W} \cap \overline{V} = \emptyset$ if $V_k = (b^f_k, 1]$. In the case (ii) we get $\overline{W} \subseteq V$ if $V_k = (b^f_k, 1]$ and $\overline{W} \cap \overline{V} = \emptyset$ if $V_k = [0, a^f_k)$. In the cases (iii) and (iv) we get $\overline{W} \cap \overline{V} = \emptyset$. □

Since $\mathcal{T}^f$ is a tame family consisting of tame open balls in the Hilbert cube $\mathbb{I}^\omega$, the set $T^f = \bigcup^\infty \mathcal{T}^f$ is a tame G_δ-set in $\mathbb{I}^\omega$ by Proposition 5.1.

For each point $x = (x_i)_{i\in\omega} \in G^\omega$, consider the function $f_x : \omega \to \omega$ assigning to each number $n \in \omega$ the smallest number $m = f_x(n)$ such that $x_0, \dots, x_n \in \bigcup_{m\leq f_x(n)} U_{n,f_x(n)}$.

Claim 5.8. *For a function $f \in \omega^\omega$ and a point $x \in G^\omega$ with $f \not\leq^* f_x$ we get $x \in T^f$.*

Proof. It follows that for every $n \in \omega$ with $f_x(n) < f(n)$, we get

$$x_0, \dots, x_n \in \bigcup_{m\leq f_x(n)} U_{n,m} \subseteq \bigcup_{m\leq f(n)} U_{n,m},$$

which implies that $x = (x_i)_{i\in\omega} \in \bigcup \mathcal{T}^f_n$ and hence $x \in \bigcup^\infty \mathcal{T}^f = T^f$ as the set $\{n \in \omega : f_x(n) < f(n)\}$ is infinite. □

Now we can complete the proof of Proposition 5.5.

1. By the definition of the cardinal $\mathfrak{b}$, there is a subset $\mathcal{F} \subseteq \omega^\omega$ of cardinality $|\mathcal{F}| = \mathfrak{b}$ such that for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ with $f \not\leq^* g$. Then for any point $x \in G^\omega$ there is a function $f \in \mathcal{F}$ such that $f \not\leq^* f_x$. By Claim 5.8, $x \in T^f \subseteq \bigcup_{g\in\mathcal{F}} T^g$, which means that G^ω is covered by $\mathfrak{b}$ many tame G_δ-sets T^g, $g \in \mathcal{F}$, in $\mathbb{I}^\omega$.

2. If $X \subseteq G^\omega$ is a subset of cardinality $|X| < \mathfrak{d}$, then the set $\{f_x : x \in X\}$ is not dominating in ω^ω and hence there is a function $f \in \omega^\omega$ such that $f \not\leq^* f_x$ for all $x \in X$. By Claim 5.8, $x \in T^f$ for each $x \in X$, which means that X is covered by the tame G_δ-set T^f. □

The following two lemmas imply Theorem 1.6.

Lemma 5.9. $\operatorname{cov}(\sigma\mathcal{G}_0) \leq \operatorname{add}(\mathcal{M})$.

Proof. Since $\operatorname{add}(\mathcal{M}) = \min\{\operatorname{cov}(\mathcal{M}), \mathfrak{b}\}$, it suffices to prove that $\operatorname{cov}(\sigma\mathcal{G}_0) \leq \min\{\operatorname{cov}(\mathcal{M}), \mathfrak{b}\}$. The inequality $\operatorname{cov}(\sigma\mathcal{G}_0) \leq \operatorname{cov}(\mathcal{M})$ trivially follows from the inclusion $\sigma\mathcal{C}_0 \subseteq \sigma\mathcal{G}_0$ and the equality $\operatorname{cov}(\sigma\mathcal{C}_0) = \operatorname{cov}(\mathcal{M})$ proved in Lemma 4.4.

To prove that $\operatorname{cov}(\sigma\mathcal{G}_0) \leq \mathfrak{b}$, apply Lemma 5.4 and find an uncountable family $(G_\alpha)_{\alpha\in\omega_1}$ of dense tame G_δ-sets in $\mathbb{I}$ such that each countable subset of $\mathbb{I}$ is contained in some G_α. This implies that $\mathbb{I}^\omega = \bigcup_{\alpha\in\omega_1} G^\omega_\alpha$.

By Proposition 5.5(1), each set G^ω_α can be covered by $\mathfrak b$ tame G_δ-subsets of the Hilbert cube. Consequently, $\mathbb I^\omega$ can be covered by $\omega_1\times\mathfrak b=\mathfrak b$ tame G_δ-subsets of the Hilbert cube, which means that $\mathrm{cov}(\sigma\mathcal G_0)\le\mathfrak b$. □

Lemma 5.10. $\mathrm{non}(\sigma\mathcal G_0)\ge\mathrm{cof}(\mathcal M)$.

Proof. Since $\mathrm{cof}(\mathcal M)=\max\{\mathrm{non}(\mathcal M),\mathfrak d\}$, it suffices to prove that $\mathrm{non}(\sigma\mathcal G_0)\ge\max\{\mathrm{non}(\mathcal M),\mathfrak d\}$. Then inequality $\mathrm{non}(\sigma\mathcal G_0)\ge\mathrm{non}(\mathcal M)$ trivially follow from the inclusion $\sigma\mathcal C_0\subseteq\sigma\mathcal G_0$ and the equality $\mathrm{non}(\sigma\mathcal C_0)=\mathrm{non}(\mathcal M)$ proved in Lemma 4.5.

To prove that $\mathrm{non}(\sigma\mathcal G_0)\ge\mathfrak d$, fix any subset $X\subseteq\mathbb I^\omega$ of cardinality $|A|<\mathfrak d$. Then $X\subseteq Y^\omega$ for some subset $Y\subseteq\mathbb I$ of cardinality $|Y|\le\aleph_0\cdot|X|<\mathfrak d\le\mathfrak c$. By Lemma 5.4, Y is contained in some dense tame G_δ-set $G\subseteq\mathbb I$. By Proposition 5.5(2), the set $X\subseteq G^\omega$ can be covered by a tame G_δ-subset of the Hilbert cube $\mathbb I^\omega$, which implies that $\mathrm{non}(\sigma\mathcal G_0)\ge\mathrm{non}(\mathcal G_0)\ge\mathfrak d$. □

6. Proof of Theorem 1.10

Assume that $\mathcal I$ is a non-trivial σG^*_δ-generated topologically invariant σ-ideal $\mathcal I$ on $\mathbb I^\omega$. By Theorem 1.1, $\mathcal I\subseteq\mathcal M$ and by Corollary 1.7, $\mathrm{add}(\mathcal I)\le\mathrm{add}(\mathcal M)$, $\mathrm{cov}(\mathcal I)=\mathrm{cov}(\mathcal M)$, $\mathrm{non}(\mathcal I)=\mathrm{non}(\mathcal M)$, and $\mathrm{cof}(\mathcal I)\ge\mathrm{cof}(\mathcal M)$. So, it remains to check that $\mathrm{add}(\mathcal I)\ge\mathrm{add}(\mathcal M)$ and $\mathrm{cof}(\mathcal I)\le\mathrm{cof}(\mathcal M)$.

Since the ideal $\mathcal I$ is σG^*_δ-generated, there is a sequence $(\mathcal I_n)_{n\in\omega}$ of G^*_δ-generated ideals such that $\bigcup_{n\in\omega}\mathcal I_n\subseteq\mathcal I$ and each set $A\in\mathcal I$ is contained in the union $\bigcup_{n\in\omega}A_n$ of some compact sets $A_n\in\mathcal I_n$, $n\in\omega$. By Definition 1.9, for every $n\in\omega$ there exist a G_δ-generated ideal $\mathcal I^*_n$ on $\mathbb I^\omega$ and an embedding $e_n:\mathbb I^\omega\to\mathbb I^\omega$ such that $\mathcal I_n=\{e_n^{-1}(A):A\in\mathcal I^*_n\}$. Since $\mathcal I^*_n$ is G_δ-generated, there is a dense G_δ-set $G_n\subseteq\mathbb I^\omega$ such that each compact subset of G belongs to the ideal $\mathcal I^*_n$ and for each compact set $A\in\mathcal I^*_n$ the G_δ-set $\mathcal H^{G_n}_A=\{h\in\mathcal H(\mathbb I^\omega):h(A)\subseteq G_n\}$ is dense in the homeomorphism group $\mathcal H(\mathbb I^\omega)$.

To show that $\mathrm{add}(\mathcal I)\ge\mathrm{add}(\mathcal M)$, it suffices to check that for each family $\mathcal A\subseteq\mathcal I$ of cardinality $|\mathcal A|<\mathrm{add}(\mathcal M)$ the union $\bigcup\mathcal A$ belongs to the ideal $\mathcal I$. For each set $A\in\mathcal A$ we can find a sequence of compact sets $A_n\in\mathcal I_n$, $n\in\omega$, whose union $\bigcup_{n\in\omega}A_n$ contains the set A. It follows that the set $e_n(A_n)$ belongs to the ideal $\mathcal I^*_n$. The choice of the G_δ-set G_n guarantees that the G_δ-set $\mathcal H^{G_n}_{e_n(A_n)}=\{h\in\mathcal H(\mathbb I^\omega):h(e_n(A_n))\subseteq G_n\}$ is dense in the homeomorphism group $\mathcal H(\mathbb I^\omega)$.

Since $|\mathcal A|<\mathrm{add}(\mathcal M)\le\mathrm{cov}(\mathcal M)$, the intersection $\mathcal H_n=\bigcap_{A\in\mathcal A}\mathcal H^{G_n}_{e_n(A_n)}$ is not empty and hence contains some homeomorphism $h_n\in\mathcal H(\mathbb I^\omega)$. Since the ideal $\mathcal I$ is non-trivial, each dense G_δ-set G_n is not equal to $\mathbb I^\omega$, which implies that the countable product $G=\prod_{n\in\omega}G_n$ is nowhere locally compact. Consequently, we can write the Polish nowhere locally compact space G as a perfect image of the Baire space ω^ω and use this fact to prove that $\mathrm{add}(\sigma\mathcal K(G))=\mathrm{add}(\sigma\mathcal K)=\mathfrak b$ and $\mathrm{cof}(\sigma\mathcal K(G))=\mathrm{cof}(\sigma\mathcal K)=\mathfrak d$.

For each set $A\in\mathcal A$ consider the compact set $K_A=\prod_{n\in\omega}h_n\circ e_n(A_n)\subseteq\prod_{n\in\omega}G_n=G$. Since $|\mathcal A|<\mathrm{add}(\mathcal M)\le\mathfrak b=\mathrm{add}(\sigma\mathcal K(G))$, there is a σ-compact set $K\subseteq G$ containing the union $\bigcup_{A\in\mathcal A}K_A$. For every $n\in\omega$ let $K_n\subseteq G_n$ be the projection of the σ-compact set $K\subseteq\prod_{i\in\omega}G_i$ onto the nth coordinate. It follows that K_n is a σ-compact subset of G_n containing the union $\bigcup_{A\in\mathcal A}h_n\circ e_n(A_n)$. Then $h_n^{-1}(K_n)\in\mathcal I^*_n$ is a σ-compact subset of $\mathbb I^\omega$ containing the union $\bigcup_{A\in\mathcal A}e_n(A_n)$ and $U_n=e_n^{-1}(h_n^{-1}(K_n))\in\mathcal I_n$ is a σ-compact set containing the union $\bigcup_{A\in\mathcal A}A_n$. Finally, the σ-compact set $U=\bigcup_{n\in\omega}U_n\in\mathcal I$ contains the union $\bigcup_{n\in\omega}\bigcup_{A\in\mathcal A}A_n\supset\bigcup\mathcal A$. This completes the proof of the inequality $\mathrm{add}(\mathcal I)\ge\mathrm{add}(\mathcal M)$.

Next, we show that $\mathrm{cof}(\mathcal I)\le\mathrm{cof}(\mathcal M)$. In the Polish group $\mathcal H(\mathbb I^\omega)^\omega$ fix a non-meager subset $H\subseteq\mathcal H(\mathbb I^\omega)^\omega$ of cardinality $|H|=\mathrm{non}(\mathcal M)$. As we already know the σ-ideal $\sigma\mathcal K(G)$ generated by compact subsets of the Polish nowhere locally compact space $G=\prod_{n\in\omega}G_n$ has cofinality $\mathrm{cof}(\sigma\mathcal K(G))=\mathrm{cof}(\sigma\mathcal K)=\mathfrak d$. Consequently, $\sigma\mathcal K(G)$ has a base $\mathcal D$ of cardinality $|\mathcal D|=\mathfrak d$ consisting of σ-compact subsets of G. For each set $D\in\mathcal D$ and $n\in\omega$ by $D_n\subseteq G_n$ denote the projection of $D\subseteq G=\prod_{i\in\omega}G_i$ onto the n-th coordinate. Observe that D_n is a σ-compact subset of G_n and hence D_n belongs to the ideal $\mathcal I^*_n$.

It follows that the family

$$\mathcal B=\Big\{\bigcup_{n\in\omega}(h_n\circ e_n)^{-1}(D_n):D\in\mathcal D,\ (h_n)_{n\in\omega}\in H\Big\}$$

consists of σ-compact subsets of $\mathbb I^\omega$, belongs to the σ-ideal $\mathcal I$ and has cardinality

$$|\mathcal B|\le|\mathcal D|\cdot|H|=\max\{\mathfrak d,\mathrm{non}(\mathcal M)\}=\mathrm{cof}(\mathcal M).$$

It remains to check that $\mathcal{B}$ is a base for the ideal $\mathcal{I}$. Fix any set $A \in \mathcal{I}$ and find compact sets $A_n \in \mathcal{I}_n$, $n \in \omega$, whose union $\bigcup_{n\in\omega} A_n$ contains A. For every $n \in \omega$ consider the compact set $e_n(A_n) \in \mathcal{I}_n^*$ and observe that the set $\mathcal{H}^{G_n}_{e_n(A_n)} = \{h \in \mathcal{H}(\mathbb{I}^\omega) : h(e_n(A_n)) \subseteq G_n\}$ is a dense G_δ in the homeomorphism group $\mathcal{H}(\mathbb{I}^\omega)$. Then the product $\prod_{n\in\omega} \mathcal{H}^{G_n}_{e_n(A_n)}$, being a dense G_δ-set in the Polish space $\mathcal{H}(\mathbb{I}^\omega)^\omega$, has a common point $(h_n)_{n\in\omega}$ with the non-meager set H. By the choice of the family $\mathcal{D}$, the compact subset $\prod_{n\in\omega} h_n \circ e_n(A_n) \subseteq G$ is contained in some set $D \in \mathcal{D}$. Then $h_n \circ e_n(A_n) \subseteq D_n$ for every $n \in \omega$ and the set $\bigcup_{n\in\omega}(h_n \circ e_n)^{-1}(D_n) \in \mathcal{B}$ contains the union $\bigcup_{n\in\omega} A_n \supseteq A$, witnessing that $\mathrm{cof}(\mathcal{I}) \le |\mathcal{B}| \le \mathrm{cof}(\mathcal{M})$.

7. Proof of Theorem 1.12

(1) We shall prove that for every $n \le \omega$ the σ-ideal $\sigma\mathcal{Z}_n$ generated by Z_n-sets in $\mathbb{I}^\omega$ is G_δ-generated. By Theorem 2.3, the set of Z-embeddings is dense in the function space $C(\mathbb{I}^n, \mathbb{I}^\omega)$. This fact allows us to construct a dense countable subset $\{f_k\}_{k\in\omega} \subseteq C(\mathbb{I}^n, \mathbb{I}^\omega)$ consisting of Z-embeddings with pairwise disjoint images. We claim that the G_δ-set $G = \mathbb{I}^\omega \setminus \bigcup_{k\in\omega} f_n(\mathbb{I}^n)$ witnesses that the ideal $\sigma\mathcal{Z}_n$ is G_δ-generated. It is clear that the ideal $\sigma\mathcal{Z}_n$ has σ-compact base. Since the set $\{f \in C(\mathbb{I}^n, \mathbb{I}^\omega) : f(\mathbb{I}^n) \cap G = \emptyset\}$ is dense in $C(\mathbb{I}^n, \mathbb{I}^\omega)$, each compact subset of G is a Z_n-set in $\mathbb{I}^\omega$. Finally, take any Z_n-set $A \subseteq \mathbb{I}^\omega$. Then we can find a dense subset $\{g_k\}_{k\in\omega} \subseteq C(\mathbb{I}^n, \mathbb{I}^\omega)$ consisting of Z-embeddings with pairwise disjoint images such that $A \cap \bigcup_{k\in\omega} g_k(\mathbb{I}^n) = \emptyset$. Using Theorems 5.1 and 11.1 of [8], by the standard back-and-forth argument we can show that the set $H = \{h \in \mathcal{H}(\mathbb{I}^\omega) : h(\bigcup_{k\in\omega} g_k(\mathbb{I}^n)) = \bigcup_{k\in\omega} f_k(\mathbb{I}^n)\}$ is dense in $\mathcal{H}(\mathbb{I}^\omega)$. Then the set $\mathcal{H}^G_A = \{h \in \mathcal{H}(\mathbb{I}^\omega) : h(A) \subseteq G\} \supseteq H$ also is dense in $\mathcal{H}(\mathbb{I}^\omega)$. Therefore, the σ-ideal $\sigma\mathcal{Z}_n$ is G_δ-generated.

(2) Let $\mathcal{D}$ be a dimension class. By definition, $\mathcal{D}$ contains a $\mathcal{K} \cap \mathcal{D}$-universal space $U \in \mathcal{D}$. Replacing U by a suitable topological copy, we can assume that U is contained in some Z-set $Z \subseteq \mathbb{I}^\omega$. Using Theorem 11.2 of [8], we can construct a countable dense subset $\{h_n\}_{n\in\omega} \subseteq \mathcal{H}(\mathbb{I}^\omega)$ such that $h_n(Z) \cap h_m(Z) = \emptyset$ for any distinct numbers $n, m \in \omega$. Taking into account that $\bigcup_{n\in\omega} h_n(Z)$ is a σZ-set in $\mathbb{I}^\omega$, we can find a countable dense subset $\{f_n\}_{n\in\omega} \subseteq C(\mathbb{I}^\omega, \mathbb{I}^\omega)$ such that $\bigcup_{n\in\omega} f_n(\mathbb{I}^\omega) \cap \bigcup_{n\in\omega} h_n(Z) = \emptyset$.

Since $\{h_n(U)\}_{n\in\omega} \subseteq \mathcal{D}$ is a closed countable cover of the set $W = \bigcup_{n\in\omega} h_n(U)$, the set W belongs to the dimension class $\mathcal{D}$. By Definition 1.11, the set $W \in \mathcal{D}$ can be enlarged to a G_δ-set $G \in \mathcal{D}$. We can additionally assume that $G \subseteq \mathbb{I}^\omega \setminus \bigcup_{n\in\omega} f_n(\mathbb{I}^\omega)$. We claim that the G_δ-set G witnesses that the σ-ideal $\sigma(\mathcal{Z}_\omega \cap \mathcal{D})$ is G_δ-generated. It is clear that this ideal has σ-compact base.

Since the set $\{f_n\}_{n\in\omega}$ is dense in the function space $C(\mathbb{I}^\omega, \mathbb{I}^\omega)$ and the set G is disjoint with the union $\bigcup_{n\in\omega} f_n(\mathbb{I}^\omega)$, each compact subset belongs to the family $\mathcal{Z}_\omega \cap \mathcal{D}$. It remains to prove that for each compact subset $A \in \sigma(\mathcal{Z}_\omega \cap \mathcal{D})$ the set $\mathcal{H}^G_A = \{h \in \mathcal{H}(\mathbb{I}^\omega) : h(A) \subseteq G\}$ is dense in the homeomorphism group $\mathcal{H}(\mathbb{I}^\omega)$. The additivity property of the dimension class $\mathcal{D}$ implies that A belongs to the class $\mathcal{D}$. By Theorem 3.1(3) of [8], A is a Z_ω-set in $\mathbb{I}^\omega$. Since the set U is $\mathcal{K} \cap \mathcal{D}$-universal, there is an embedding $h : A \to U \subseteq Z$. By the Z-Set Unknotting Theorem 11.2 [8], the homeomorphism $h : A \to h(A)$ between the Z_ω-sets A and $h(A)$ can be extended to a homeomorphism $\bar{h}$ of $\mathbb{I}^\omega$.

To show that the set $\mathcal{H}^G_A$ is dense in $\mathcal{H}(\mathbb{I}^\omega)$, fix any homeomorphism $g \in \mathcal{H}(\mathbb{I}^\omega)$ and an open neighborhood $O(g)$ of g in $\mathcal{H}(\mathbb{I}^\omega)$. By the density of the set $\{h_n\}_{n\in\omega}$ in $\mathcal{H}(\mathbb{I}^\omega)$, for some $n \in \omega$ the homeomorphism $h_n \circ \bar{h}$ belongs to the open set $O(g)$. Since $h_n \circ \bar{h}(A) \subseteq h_n(U) \subseteq W \subseteq G$, the homeomorphism $h_n \circ \bar{h}$ belongs to $O(g) \cap \mathcal{H}^G_A$ witnessing that the set $\mathcal{H}^G_A$ is dense in $\mathcal{H}(\mathbb{I}^\omega)$. This completes the proof of the G_δ-generacy of the ideal $\sigma(\mathcal{Z}_\omega \cap \mathcal{D})$.

Next, we show that the ideal $\sigma(\mathcal{K} \cap \mathcal{D})$ is G^*_δ-generated. For this take any Z-embedding $e : \mathbb{I}^\omega \to \mathbb{I}^\omega$ and observe that $\sigma(\mathcal{K} \cap \mathcal{D}) = \{e^{-1}(A) : A \in \sigma(\mathcal{Z}_\omega \cap \mathcal{D})\}$. Since the ideal $\sigma(\mathcal{Z}_\omega \cap \mathcal{D})$ is G_δ-generated, the σ-ideal $\sigma(\mathcal{K} \cap \mathcal{D})$ is G^*_δ-generated.

(3) Let $\mathcal{D}$ be a σ-dimensional class. Then $\mathcal{D} = \bigcup_{n\in\omega} \mathcal{D}_n$ for some increasing family of dimensional classes $(\mathcal{D}_n)_{n\in\omega}$. By (already proved) Theorem 1.12(2), the σ-ideals $\sigma(\mathcal{Z}_\omega \cap \mathcal{D}_n)$ and $\sigma(\mathcal{K} \cap \mathcal{D}_n)$ are G^*_δ-generated for all $n \in \omega$.

To show that the σ-ideal $\sigma(\mathcal{Z}_\omega \cap \mathcal{D})$ is σG^*_δ-generated, fix any subset $A \in \sigma(\mathcal{Z}_\omega \cap \mathcal{D})$. By the definition of the σ-ideal $\sigma(\mathcal{Z}_\omega \cap \mathcal{D})$, the set A is contained in the union $\bigcup_{k\in\omega} A_k$ of some Z_ω-sets $A_k \in \mathcal{D}$. For every $n \in \omega$ let $B_n = \bigcup\{A_k : k \le n,\ A_k \in \mathcal{D}_n\} \in \mathcal{Z}_\omega \cap \mathcal{D}_n$. Since $A \subseteq \bigcup_{k\in\omega} A_k = \bigcup_{n\in\omega} B_n$, we see that the σ-ideal $\sigma(\mathcal{Z}_\omega \cap \mathcal{D})$ is σG^*_δ-generated. By analogy we can prove the σG^*_δ-generacy of the σ-ideal $\sigma(\mathcal{K} \cap \mathcal{D}_n)$.

8. Open Problems

In this section we collect some open problems on topologically invariant σ-ideals on $\mathbb{I}^\omega$. The most intriguing problems concern the σ-ideal $\sigma\mathcal{G}_0$.

Problem 8.1. *Is* $\mathrm{add}(\sigma\mathcal{G}_0) = \mathrm{cov}(\sigma\mathcal{G}_0) = \omega_1$ *and* $\mathrm{non}(\sigma\mathcal{G}_0) = \mathrm{cof}(\sigma\mathcal{G}_0) = \mathfrak{c}$*?*

Problem 8.2. *Is* $\mathrm{cov}(\sigma\mathcal{G}_0) = \mathfrak{c}$ *under Martin's Axiom? Under PFA?*

Problem 8.3. *Is* $\sigma\mathcal{G}_0 = \sigma\mathcal{D}_0$*?*

It Proposition 4 of [4] it was proved that for any dense G_δ-set $G \subseteq \mathbb{I}$ the countable power G^ω does not belong to the family $\mathcal{G}_0$ of minimal dense G_δ-sets in $\mathbb{I}^\omega$.

Problem 8.4. *Is* $G^\omega \in \sigma\mathcal{G}_0$ *for some dense* G_δ*-set* $G \subseteq \mathbb{I}$*?*

Problem 8.5. *Let* $\mathcal{I}$ *be a maximal non-trivial topologically invariant* σ*-ideal with Borel base on* $\mathbb{I}^\omega$*. Is* $\mathcal{I} = \mathcal{M}$*?*

A closed subset $A \subseteq \mathbb{I}^\omega$ is called a *homological* Z_ω*-set* in $\mathbb{I}^\omega$ if $A \times \{0\}$ is a Z_ω-set in $\mathbb{I}^\omega \times [-1, 1]$. Let $\mathcal{B}$ be the family of all Borel subsets $B \subseteq \mathbb{I}^\omega$ such that the closure $\bar{C}$ of each connected subset $C \subseteq B$ in $\mathbb{I}^\omega$ is a homological Z_ω-set in $\mathbb{I}^\omega$. It follows from Main Lemma of [1] that the σ-ideal $\sigma\mathcal{B}$ generated by the family $\mathcal{B}$ is non-trivial.

Problem 8.6. *Is the ideal* $\sigma\mathcal{B}$ *a maximal non-trivial topologically invariant* σ*-ideal with Borel base on* $\mathbb{I}^\omega$*?*

References

[1] T. Banakh, R. Cauty, *On universality of countable and weak products of sigma hereditarily disconnected spaces*, Fund. Math. **167** (2001) 97–109.
[2] T. Banakh, R. Cauty, M. Zarichnyi, *Open problems in infinite-dimensional topology*, in: Open Problems in Topology, II (E.Pearl ed.), Elsevier, 2007, p.601–624.
[3] T. Banakh, M. Morayne, R. Rałowski, S. Żeberski, *Topologically invariant σ-ideals in Euclidean spaces*, preprint (http://arxiv.org/abs/1208.4823).
[4] T. Banakh, D. Repovš, *Universal meager F_σ-sets in locally compact manifolds*, Comment. Math. Univ. Carolin. (to appear); available at: http://arxiv.org/abs/1302.5653
[5] T. Bartoszyński, H. Judah, *Set theory: on the structure of the real line*, Wellesley, MA: A. K. Peters Ltd., 1995.
[6] T. Bartoszyński, H. Judah, S. Shelah, *The Cichon diagram*, J. Symbolic Logic **58**:2 (1993), 401–423.
[7] A. Blass, *Combinatorial cardinal characteristics of the continuum*,in: Handbook of set theory, 395–489, Springer, Dordrecht, 2010.
[8] T.A. Chapman, *Lectures on Hilbert cube manifolds*, Amer. Math. Soc., Providence, R. I., 1976.
[9] E. van Douwen, *The integers and topology*, in: Handbook of set-theoretic topology, 111–167, North-Holland, Amsterdam, 1984.
[10] A.N. Dranishnikov, *On the cohomological dimension of compacta*, Trudy Mat. Inst. Steklov. **193** (1992), 115–120; Engl. transl. in Proc. Steklov Inst. Math. **193**:3 (1993) 125–130.
[11] A. Dranishnikov, *Cohomological dimension theory of compact metric spaces*, Topological Atlas Invited Contributions, **6** (2001), 7–73; (http://at.yorku.ca/t/a/i/c/43.htm).
[12] A. Dranishnikov, J. Dydak, *Extension theory of separable metrizable spaces with applications to dimension theory*, Trans. Amer. Math. Soc. **353**:1 (2001), 133–156.
[13] J. Dydak. *Cohomological dimension theory*, in: Handbook of Geometric Topology (R.Daverman, R.Sher, eds.), Elsevier, 2002; p.423–470.
[14] R. Engelking, *Theory of dimensions finite and infinite*, Heldermann Verlag, Lemgo, 1995.
[15] A. Kechris, *Classical descriptive set theory*, Springer, NY, 1995.
[16] N. Kroonenberg, *Characterization of finite-dimensional Z-sets*. Proc. Amer. Math. Soc. **43** (1974), 421–427.
[17] J. van Mill, *Infinite-dimensional topology. Prerequisites and introduction*, North-Holland Publ., Amsterdam, 1989.
[18] J. Vaughan, *Small uncountable cardinals and topology*, Open problems in topology, 195–218, North-Holland, Amsterdam, 1990.

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