# On weakly measurable functions

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ABSTRACT. We show that if $T$ is uncountable Polish space, $\mathcal{X}$ is metrizable space and $f : T \to \mathcal{X}$ is weakly Baire measurable function, then we can find a meagre set $M \subseteq T$ such that $f[T \setminus M]$ is separable space. We also give an example showing that the assumption that "$\mathcal{X}$ is metrizable" cannot be replaced by the condition "$\mathcal{X}$ is normal".

## 1. Introduction

It is known that for a partition $\mathcal{A}$ of the real line consisting of sets of Lebesgue measure zero, the union of some of these sets is Lebesgue nonmeasurable. Analogous result is known for the sets of the first category (the Lebesgue measurability is then replaced by having the Baire property). Actually, this result remains true, if in the above statement, the real line is replaced by any Polish space, the $\sigma$-ideals of sets of Lebesgue measure zero or of sets of the first category are replaced by any $\sigma$-ideal $I$ with a Borel base and instead of assuming that the family $\mathcal{A}$ of sets of an ideal is a partition of the space, we assume that $\mathcal{A}$ is point finite and its union is not in $I$. The conclusion says now that there exists a subfamily $\mathcal{A}'$ of $\mathcal{A}$ such that the union of its sets is not in the $\sigma$-algebra generated by the $\sigma$-algebra of Borel sets and $I$ (see [**1**]). This result is called in the literature Four Poles Theorem.

It is known that within ZFC it not possible to replace the assumption that the family $\mathcal{A}$ is point-finite even by the one saying that $\mathcal{A}$ is point-countable (see [**5**] p. 64).

In various cases it is possible to obtain more than nonmeasurability of the union of a subfamily of $\mathcal{A}$. Namely, the intersection of this union with any measurable set that is not in $I$ is nonmeasurable (recall, the

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measurability is understood here in the sense of belonging to the $\sigma$-algebra generated by the family of Borel sets and $I$). Such strong conclusion can be obtained for the ideal of first Baire category sets under the assumption that $\mathcal{A}$ is a partition, but without assuming anything about the regularity of the elements of $\mathcal{A}$ (see [**3**]).

Four Poles Theorem gives a variety of applications. Many of them can be found in [**2**]. Let us recall that the function $f : \mathbb{R} \to \mathcal{X}$ is called Baire measurable if for every open set $U \subseteq \mathcal{X}$ the set $f^{-1}[U]$ has Baire property. In the similar way we can define $\mathcal{B}[I]$-measurable function (for $I$ as in Four Poles Theorem). One of the results from [**2**] says that if $\mathcal{X}$ is metrizable space and $f : \mathbb{R} \to \mathcal{X}$ is $\mathcal{B}[I]$-measurable function, then we can find a set $A \in I$ such that $f[\mathbb{R} \setminus A]$ is separable.

In this paper we show similar result for wilder class of functions. The proof of this result uses the result from [**3**]. We also show that in our result the assumption that "$\mathcal{X}$ is metrizable" cannot be replaced by the condition "$\mathcal{X}$ is normal".

## 2. Definitions and notations

The cardinality of a set $A$ is denoted by $|A|$. Cardinal numbers will usually be denoted by $\kappa$ and $\lambda$. The family of all subsets of cardinality not bigger than $\kappa$ of a set $A$ is denoted by $[A]^{\leq\kappa}$. The set of real numbers is denoted by $\mathbb{R}$. An *ideal* $I$ of subsets of a set $X$ is a family of subsets of $X$ which is closed under finite unions and taking subsets and such that $[X]^{<\omega} \subseteq I$. A family of sets is a *$\sigma$-ideal* if it is an ideal and is closed under countable unions. For an ideal $I$, a symbol $I^*$ denotes its dual filter.

For a topological space $T$, by $\mathcal{O}_T$ we denote the family of all open subsets of T and by $\mathcal{B}_T$ we denote the family of Borel subsets of $T$. If $I$ is an ideal of subsets of a set $X$ and $\mathcal{S}$ is a field of subsets of $X$, then by $\mathcal{S}[I]$ we denote the field generated by $\mathcal{S} \cup I$. If $I$ is a $\sigma$-ideal and $\mathcal{S}$ is a $\sigma$-field then $\mathcal{S}[I]$ is a $\sigma$-field, too.

Let $T$ be an uncountable Polish topological space. The $\sigma$-ideal of all meagre sets of $T$ is denoted by $\mathbb{K}_T$. The $\sigma$-ideal of Lebesgue measure zero subsets of $\mathbb{R}$ will be denoted by $\mathbb{L}$ and the $\sigma$-ideal of sets of the first Baire category in $\mathbb{R}$ will be denoted by $\mathbb{K}$. Then $B_\mathbb{R}[\mathbb{L}]$ is the $\sigma$-field of Lebesgue measurable subsets of $\mathbb{R}$ and $B_\mathbb{R}[\mathbb{K}]$ is the $\sigma$-field of subsets of $\mathbb{R}$ with the Baire property.

If $I$ is an ideal of subsets of a topological space $T$ then we say that the ideal $I$ has a Borel base if for each set $X \in I$ there exists a set $Y \in \mathcal{B}_T$ such that $X \subseteq Y$. The two classical ideals $\mathbb{K}$ and $\mathbb{L}$ have Borel bases.

Let $\mathbb{B}$ be a complete Boolean algebra. We say that $\mathbb{B}$ satisfies c.c.c. (countable chain condition) if every antichain of elements of $\mathbb{B}$ is countable.

For an infinite cardinal $\kappa$ let us consider the topological sum of the family of spaces $\{[0,1]\times\{\xi\} : \xi < \kappa\}$. Let us identify in this sum all points $(0,\xi)$, where $\xi < \kappa$, and denotethe obtained space by $J_\kappa$. It is well known that any metrizable space can be embeded into the countable product of spaces $J_{\kappa_n}$ (see [**4**]).

If $\mathcal{X}$ is metrizable space and $\mathcal{A}$ is an open cover of $\mathcal{X}$ then, by Stone's theorem we can find a family $\{\mathcal{A}_n\}_{n\in\omega}$ such that $\bigcup_{n\in\omega}\mathcal{A}_n$ is a point-finite open cover and each $\mathcal{A}_n$ is a family of pairwise disjoint open sets and each element of $\mathcal{A}_n$ is contained in some element of $\mathcal{A}$. This result can be found in [**4**].

## 3. Weakly measurable functions

In this section $T$ denote an uncountable Polish space. Let us recall the following definition.

Definition 3.1. Let $N \subseteq X \subseteq T$. We say that the set $N$ is *completely Baire nonmeasurable* in $X$ if

$$(\forall A \in \mathcal{B}_T)(A\cap X \notin \mathbb{K}_T \to (A\cap N \notin \mathbb{K}_T)\wedge (A\cap (X\setminus N)\notin \mathbb{K}_T)).$$

In the paper [**3**] we can find the following theorem.

Theorem 3.2. *Let $\mathcal{P}$ be a pairwise disjoint family of meagre sets such that $\bigcup\mathcal{P}\notin\mathbb{K}_T$. Then there exists a subfamily $\mathcal{P}'\subseteq\mathcal{P}$ such that $\bigcup\mathcal{P}'$ is completely Baire nonmeasurable in $\bigcup\mathcal{P}$.*

Definition 3.3. Let $I\subseteq P(T)$ be a $\sigma$-ideal with Borel base such that the Boolean algebra $\mathcal{B}_T/I$ satisfies c.c.c. Let $A$ be any subset of $T$. By $[A]_I$ we denote the Borel envelope of A, i.e. the minimal (in sense of the algebra $\mathcal{B}_T/I$) Borel set containing A.

The set $[A]_I$ is well defined since the algebra $\mathcal{B}_T/I$ satisfies c.c.c. To find $[A]_I$, take the maximal antichain $\mathcal{A}$ of $I$-positive Borel sets which are disjoint from $A$. The family $\mathcal{A}$ is countable. So, its union $\bigcup\mathcal{A}$ is a Borel set. The complement $(\bigcup\mathcal{A})^c$ is the required envelope.

Definition 3.4. Let $\mathcal{X}$ be a topological space and $f : T\to\mathcal{X}$ be a function. We say that $f$ is *weakly Baire measurable* if

$$(\forall U,V\in\mathcal{O}_{\mathcal{X}})(U\cap V=\emptyset\wedge f^{-1}[U]\notin\mathbb{K}_T\wedge f^{-1}[V]\notin\mathbb{K}_T\to$$

$$\to [f^{-1}[U]]_{\mathbb{K}_T}\neq[f^{-1}[V]]_{\mathbb{K}_T}).$$

First, let us notice that every Baire measurable function $f : T \to \mathcal{X}$ is weakly Baire measurable. Indeed, if we have two disjoint open sets $U, V \subseteq \mathcal{X}$ such that $f^{-1}[U] \notin \mathbb{K}_T$ and $f^{-1}[V] \notin \mathbb{K}_T$ then since they are disjoint and have Baire property $[f^{-1}[U]]_{\mathbb{K}_T} \cap [f^{-1}[V]]_{\mathbb{K}_T} \in \mathbb{K}_T$. It implies that the condition of weak Baire measurability is satisfied.

From the other hand, we can easily find a function $f : \mathbb{R} \to \mathbb{R}$ which is Lebesgue measurable and is not weakly Baire measurable. Namely, let us consider the following partition $\mathbb{R} = K \cup L$ such that $L \in \mathbb{L}$ and $K \in \mathbb{K}$. Let us fix any partition $L = L_1 \cup L_2$ such that $[L_1]_{\mathbb{K}} = [L_2]_{\mathbb{K}} = [\mathbb{R}]_{\mathbb{K}}$. The characteristic function of a set $L_1$ is Lebesgue measurable but it is not weakly Baire measurable.

Let us recall well known result due to Banach (see [**6**]).

Theorem 3.5 (Banach). *Let $\mathcal{X}$ be any topological space. Consider a family*

$$\mathcal{A} = \{U \in \mathcal{O}_{\mathcal{X}} : U \text{ is meagre in } \mathcal{X}\}.$$

*Then $\bigcup \mathcal{A}$ is also meagre in $\mathcal{X}$.*

The next theorem is similar to the previous one.

Theorem 3.6. *Let $\mathcal{X}$ be any metrizable space. Let $f : T \to \mathcal{X}$ be weakly Baire measurable function. Consider a family*

$$\mathcal{A} = \{f^{-1}[U] \in \mathbb{K}_T : U \in \mathcal{O}_{\mathcal{X}}\}.$$

*Then $\bigcup \mathcal{A}$ is also meagre in $T$.*

Proof. Let $\tilde{\mathcal{A}} = \{U \in \mathcal{O}_X : f^{-1}[U] \in \mathbb{K}_T\}$. We have that $\bigcup \mathcal{A} = f^{-1}[\bigcup \tilde{\mathcal{A}}]$. Since $\mathcal{X}$ is metrizable, $\bigcup \tilde{\mathcal{A}}$ is also metrizable. So, we can find families $\tilde{\mathcal{A}}_n \quad (n \in \omega)$ satisfying the following conditions:

- $\bigcup \tilde{\mathcal{A}} = \bigcup_{n\in\omega} \bigcup \tilde{\mathcal{A}}_n$,
- $\tilde{\mathcal{A}}_n$ is a family of pairwise disjoint open sets,
- $(\forall U \in \tilde{\mathcal{A}}_n)(\exists V \in \tilde{\mathcal{A}})(U \subseteq V)$.

From the above conditions follows that for each $n \in \omega$ we have that $\tilde{\mathcal{A}}_n \subseteq \tilde{\mathcal{A}}$. Define

$$\mathcal{A}_n = \{f^{-1}[U] : U \in \tilde{\mathcal{A}}_n\}.$$

We claim that $\bigcup \mathcal{A}_n \in \mathbb{K}_T$. Suppose otherwise. Then the family $\mathcal{A}_n$ satisfies the assumption in Theorem 3.2. That's why we can find two disjoint open sets $U_0$, $U_1 \subseteq \mathcal{X}$ such that $f^{-1}[U_0]$ and $f^{-1}[U_1]$ are completely Baire nonmeasurable in $\bigcup \mathcal{A}_n$. In particular, $[f^{-1}[U_0]]_{\mathbb{K}_T} = [f^{-1}[U_1]]_{\mathbb{K}_T}$, what gives a contradiction to weak Baire measurability of $f$.

Notice that $\bigcup \mathcal{A} = \bigcup_{n\in\omega} \bigcup \mathcal{A}_n$. So, $\bigcup \mathcal{A} \in \mathbb{K}_T$. □

THEOREM 3.7. *Let $I \subseteq P(T)$ be a $\sigma$-ideal with Borel base such that $\mathcal{B}_T/I$ satisfies c.c.c. Let $\{A_\xi : \xi \in \omega_1\} \subseteq P(T)$ be any family. Then we can find $B, C$ two disjoint countable subsets of $\omega_1$ such that $[\bigcup_{\xi\in B} A_\xi]_I = [\bigcup_{\xi\in C} A_\xi]_I$.*

PROOF. Fix a family $\{A_\xi : \xi \in \omega_1\} \subseteq P(T)$. For $\alpha < \beta < \omega_1$ let

$$A_\alpha^\beta = \bigcup\{A_\xi : \alpha < \xi < \beta\}.$$

Proceeding by transfinite induction, we construct a sequence $\{\alpha_\xi : \xi < \omega_1\}$ of ordinals less than $\omega_1$ such that

$$(\forall \xi, \zeta, \eta \in \omega_1)((\xi < \zeta < \eta) \to [A_{\alpha_\xi}^{\alpha_\zeta}]_I = [A_{\alpha_\xi}^{\alpha_\eta}]_I).$$

In $\beta$-step consider a sequence $\{[A_{\alpha_\beta}^{\alpha}]_I : 0 < \alpha < \omega_1\}$. It is an increasing sequence. Since $\mathcal{B}_T/I$ satisfies c.c.c. this sequence is constant from some point $\gamma$ on. Put $\alpha_{\beta+1} = \gamma$.

Now, consider the sequence $\{[A_{\alpha_\xi}^{\alpha_{\xi+1}}]_I : \xi < \omega_1\}$. It is a decreasing sequence. Since $\mathcal{B}_T/I$ satisfies c.c.c. this sequence is constant from some $\gamma < \omega_1$. Put

$$B = (\alpha_\gamma, \alpha_{\gamma+1}), \quad C = (\alpha_{\gamma+1}, \alpha_{\gamma+2}).$$

These sets satisfy required condition. □

THEOREM 3.8. *Let $\mathcal{X}$ be a metrizable space. Let $f : T \to \mathcal{X}$ be weakly Baire measurable function. Then there is a meagre set $M \subseteq T$ such that $f[T \setminus M]$ is a separable space.*

PROOF. Consider the family $\mathcal{A} = \{f^{-1}[U] \in \mathbb{K}_T : U \in \mathcal{O}_\mathcal{X}\}$. By Theorem 3.6 we have that $\bigcup \mathcal{A} \in \mathbb{K}_T$. Define $M = \bigcup \mathcal{A}$.

It is enough to show that $f[T \setminus M]$ is a separable space. The space $f[T \setminus M]$ is metrizable as a subspace of $\mathcal{X}$. So, we can treat $f[T \setminus M]$ as a subspace of $\prod_{n\in\omega} J_{\kappa_n}$. Assume that each $\kappa_n$ is the least possible cardinal.

We will show that $\kappa_n \leq \omega$. Suppose otherwise. Without loss of generality we can assume that $\kappa_0 = \omega_1$. It means that there exists a family $\{U_\alpha : \alpha < \omega_1\}$ of nonempty pairwise disjoint open sets in $f[T \setminus M]$. The corresponding family $\{f^{-1}[U_\alpha] : \alpha < \omega_1\}$ satisfies the condition $f^{-1}[U_\alpha] \notin \mathbb{K}_T$. Applying Theorem 3.7 to this family, we get a contradiction to weak measurability of $f$. So, for each $n \in \omega$ we have that $\kappa_n \leq \omega$. It means that $f[T \setminus M] \subseteq \prod_{n\in\omega} J_\omega$ what implies that $f[T \setminus M]$ is a separable space. □

## 4. Nonmetrizable spaces

Now, let us show that Theorems 3.6 and 3.8 do not hold if we replace the metrizable space $\mathcal{X}$ by a normal space. In fact, we will work with the space $\omega_1$ equipped with the order topology.

Note that we cannot hope to find a counterexample in ZFC, because in some models Four Poles Theorem or even its stronger version about completely nonmeasurable unions holds without any assumptions to the given family of small sets. In such models Theorems 3.6, 3.8 hold if we replace the metrizable space $\mathcal{X}$ by any topological space.

That's why our considerations will take place in a particular model of ZFC. Namely, let us start with a model $V$ which satisfies GCH. We can extend it, by adding $\omega_2$ random reals, to a model $V'$. The universe $V'$ is good enough for what we need. Notice that in $V'$ we have that $2^\omega = 2^{\omega_1} = \omega_2$.

Let us start with an easy observation.

Claim 4.1. *In $V'$ there exists a family $\{K^\alpha\}_{\alpha\in\omega_2} \subseteq \mathbb{K}$ such that for every uncountable set $A \subseteq \omega_2$ we have that $\bigcup_{\alpha\in A} K^\alpha = \mathbb{R}$.*

Proof. Let $\{r_\alpha\}_{\alpha\in\omega_2}$ be the family of generic random reals. Fix a partition of $\mathbb{R}$ into a $G_\delta$ set of Lebesgue measure zero and a $F_\sigma$ set of first Baire category $\mathbb{R} = K \cup L$, where $K \in \mathbb{K}$ and $L \in \mathbb{L}$. Put

$$K^\alpha = K + r_\alpha = \{k + r_\alpha : k \in K\}.$$

Let $A \subseteq \omega_2$ be uncountable. Assume that there is a real number $x \in \mathbb{R}$ satisfying $x \notin \bigcup_{\alpha\in A} K^\alpha$. So, if $\alpha \in A$ then $x \notin r_\alpha + K$. So $r_\alpha \notin x - K = \{x - k : k \in K\}$. A set $x - K$ is meager. Real number $x$ belongs to $V[r_\alpha : \alpha \in I]$, where $I$ is a countable subset of $\omega_2$. Since $A$ is uncountable, there exists $\xi \in A \setminus I$. So, $r_\xi$ is a random real over $V[r_\alpha : \alpha \in I]$. But a set $(x - K)^c \in \mathbb{L}$ and is coded in a model $V[r_\alpha : \alpha \in I]$. What is more $r_\xi \in (x - K)^c$ what gives a contradiction. $\square$

The next claim is a generalization of the result presented in [**5**] p. 64.

Claim 4.2. *In $V'$ the following statement holds:*
*"Fix any $\sigma$-ideal $\mathcal{I} \subseteq P(\omega_1)$. There exists a point-countable family $\{\tilde{K}^\alpha\}_{\alpha\in\omega_1}$ of meagre subsets of the real line such that for every set $A \subseteq \omega_1$ we have that*

$$A \in \mathcal{I} \to \bigcup_{\alpha\in A} \tilde{K}^\alpha \in \mathbb{K},$$

$$A \notin \mathcal{I} \to \bigcup_{\alpha \in A} \tilde{K}^\alpha \in \mathbb{K}^*.\text{”}$$

PROOF. Let us fix a family $\{K^\alpha\}_{\alpha\in\omega_2}$ from Claim 4.1 and fix a bijection $\phi : P(\omega_1) \to \omega_2$. Required family $\{\tilde{K}^\alpha\}_{\alpha\in\omega_1}$ will have the following properties

(1) $(\forall \alpha < \omega_1)(\tilde{K}^\alpha \subseteq K^\alpha)$,
(2) $(\forall A \in \mathcal{I}) \bigcup\{\tilde{K}^\alpha : \alpha \in A\} \subseteq K^{\phi(A)}$,
(3) $(\forall A \notin \mathcal{I}) \bigcup\{\tilde{K}^\alpha : \alpha \in A\} \cup K^{\phi(A)} = \mathbb{R}$.

To see that it is possible to construct the family satisfying the above conditions let us write them in the following form

(1) $(\forall x \in \mathbb{R})(\forall A \in \mathcal{I})(x \notin K^{\phi(A)} \to x \notin \bigcup\{\tilde{K}^\alpha : \alpha \in A\})$,
(2) $(\forall x \in \mathbb{R})(\forall A \notin \mathcal{I})(x \notin K^{\phi(A)} \to x \in \bigcup\{\tilde{K}^\alpha : \alpha \in A\})$.

Using Claim 4.1 we can deduce that for each $x \in \mathbb{R}$ there are only countably many $A \subseteq \omega_1$ such that $x \notin K^{\phi(A)}$. Let

$$\mathcal{I}(x) = \{A \in \mathcal{I} : x \notin K^{\phi(A)}\},$$

$$\mathcal{I}^+(x) = \{A \notin \mathcal{I} : x \notin K^{\phi(A)}\}.$$

Let $B(x) = \bigcup \mathcal{I}(x)$. Then $B(x) \in \mathcal{I}$ and for each $A \in \mathcal{I}^+(x)$, we have $A \setminus B(x) \notin \mathcal{I}$. In particular $A \setminus B(x) \neq \emptyset$. So, it is possible to ensure that

$$(\forall x)(\forall A \in \mathcal{I}^+(x))(\exists \alpha \in A \setminus B(x))(x \in \tilde{K}^\alpha).$$

It finishes the construction of required family. $\square$

The next result can be seen as interesting in itself.

CLAIM 4.3 (Cichoń). *In $V'$ the following statement holds: ”Fix any $\sigma$-ideal $\mathcal{I} \subseteq P(\omega_1)$. Then there exists a function $g : \mathbb{R} \to \omega_1$ such that*

$$(\forall A \subseteq \omega_1)(A \in \mathcal{I} \leftrightarrow g^{-1}[A] \in \mathbb{K}).\text{”}$$

PROOF. Let $\{\tilde{K}^\alpha\}_{\alpha\in\omega_1}$ be a family from Claim 4.2. Define

$$\tilde{K} = \bigcup\{\tilde{K}^\alpha \times \{\alpha\} : \alpha \in \omega_1\}.$$

Fix an enumeration $\{A_\alpha : \alpha < \omega_2\}$ of $\mathcal{I}$-positive subsets of $\omega_1$. Fix also an enumeration $\{M_\alpha : \alpha < \omega_2\}$ of meagre subsets of $\mathbb{R}$ and an enumeration $\{x_\alpha : \alpha < \omega_2\}$ of all real numbers. We will construct the increasing sequence $\{g_\alpha\}_{\alpha<\omega_2}$ of functions by transfinite induction. In $\alpha$-step of induction we will construct a function $g_\alpha$ satisfying the following conditions

(1) $g_\alpha \subseteq \tilde{K}$,
(2) $x_\alpha \in dom(g_\alpha)$,

(3) $\bigcup_{\xi<\alpha} g_\xi \subseteq g_\alpha$,
(4) $|g_\alpha| \leq \omega_1$,
(5) $(\forall \xi, \zeta < \alpha)(\exists x \in \mathbb{R})(x \notin M_\zeta \wedge g(x) \in A_\xi)$.

So, at $\alpha$-step, we have to add to a function $\bigcup_{\xi<\alpha} g_\xi$ at most $\omega_1$ many new points of our function $g$. Since $|\mathbb{R}| = \omega_2$, it is possible to make the construction. Put $g = \bigcup_{\alpha\in\omega_2} g_\alpha$. It is the required function. □

Now, we are able to formulate the result which shows that Theorem 3.6 cannot be generalized too much.

THEOREM 4.4. *In $V'$ the following statement holds:*
*"Let $\tau_{order}$ be the order topology on $\omega_1$. There is a function $f : \mathbb{R} \to (\omega_1, \tau_{order})$ satisfying the following conditions:*

(1) *$f$ is Baire measurable,*
(2) *$\bigcup\{f^{-1}[U] \in \mathbb{K} : U \in \tau_{order}\} = \mathbb{R}$."*

PROOF. Apply Claim 4.3 to the ideal $\mathcal{NS}_{\omega_1}$ of all nonstationary subsets of $\omega_1$. We obtain a function $f : \mathbb{R} \to \omega_1$.

This function is Baire measurable. Indeed, take any open subset $U$ of $\omega_1$. There are two possibilities.

First, $U^c$ is unbounded. Since $U^c$ is closed, $U \in \mathcal{NS}_{\omega_1}$ and $f^{-1}[U] \in \mathbb{K}$. Second, $U^c$ is bounded. Then $U^c \in \mathcal{NS}_{\omega_1}$ and $f^{-1}[U^c] \in \mathbb{K}$. So $f^{-1}[U] \in \mathbb{K}^*$. In both cases we get a set with Baire property.

Since every point in $\omega_1$ can be covered by an open set which is nonstationary, we have that $\bigcup\{f^{-1}[U] \in \mathbb{K} : U \in \tau_{order}\} = \mathbb{R}$. □

It can be easily seen that the function obtained in Theorem 4.4 is also a counterexample to a simple generalization of Theorem 3.8 for functions with images in normal topological spaces.

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