Inscribing nonmeasurable sets

Szymon Żeberski

Abstract. Our main inspiration is the work in paper [**4**]. We will prove that for a partition $\mathcal{A}$ of the real line into meager sets and for any sequence $\mathcal{A}_n$ of subsets of $\mathcal{A}$ one can find a sequence $\mathcal{B}_n$ such that $\mathcal{B}_n$'s are pairwise disjoint and have the same "outer measure with respect to the ideal of meager sets". We get also generalization of this result to a class of σ-ideals posessing Suslin property. However, in that case we use additional set-theoretical assumption about non-existing of quasi-measurable cardinal below continuum.

1. Introduction

We want to discuss analogues and strengthenings of the following result (see [**4**]).

Theorem 1.1 (Gitik, Shelah). *Let $(A_n : n \in \omega)$ be a sequence of subsets of $\mathbb{R}$. Then we can find a sequence $(B_n : n \in \omega)$ such that*

(1) $B_n \cap B_m = \emptyset$ *for* $n \neq m$,
(2) $B_n \subseteq A_n$,
(3) $\lambda^*(A_n) = \lambda^*(B_n)$, *where* λ^* *is outer Lebesgue measure.*

We obtain analogous results for the ideal of meager sets in place of null sets. Additionally, we will not work with the partition into singletons but with partition into meager sets.

We will also consider a wider class of ideals, i.e. c.c.c. ideals. In this case we will deal with point-finite families of sets from ideal instead of singletons. However, in the general case we will assume that there is no quasi-measurable cardinal smaller than continuum.

1991 *Mathematics Subject Classification.* Primary 03E35, 03E75; Secondary 28A05.

Key words and phrases. quasi-measurable cardinal, nonmeasurable set, meager set, real line, c.c.c. ideal.

We will use theorems about finding subfamilies with nonmeasurable union. At the begining let us recall the following result (see [**1**]).

Theorem 1.2 (Brzuchowski, Cichoń, Grzegorek, Ryll-Nardzewski). *Let $\mathbb{I}$ be a σ-ideal with Borel base of subsets of uncountable Polish space T. Let $\mathcal{A} \subseteq \mathbb{I}$ be a point-finite family (i.e. each $x \in T$ belongs to finitely many members of $\mathcal{A}$) such that $\bigcup \mathcal{A} = T$. Then we can find a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is $\mathbb{I}$-nonmeasurable i.e does not belong to the σ-field generated by Borel sets and ideal $\mathbb{I}$.*

In various cases it is possible to obtain more than nonmeasurability of the union of a subfamily of $\mathcal{A}$. Namely, the intersection of this union with any measurable set that is not in $\mathbb{I}$ is nonmeasurable (recall, the measurability is understood here in the sense of belonging to the σ-algebra generated by the family of Borel sets and $\mathbb{I}$). Such a strong conclusion can be obtained for the ideal of first Baire category sets under the assumption that $\mathcal{A}$ is a partition, but without assuming anything about the regularity of the elements of $\mathcal{A}$ (see [**2**]).

It is also possible to obtain complete nonmeasurability of the union of a subfamily of $\mathcal{A}$ assuming that $\mathcal{A}$ is point-finite family. The price of it is that we use some set-theoretic assumptions. Namely, we assume that there is no quasi-measurable cardinal smaller than 2^ω. (Recall that κ is quasi-measurable if there exists a κ-additive ideal I of subsets of κ such that the Boolean algebra $P(\kappa)/I$ satisfies countable chain condition.) By the Ulam theorem (see [**6**]) every quasi-measurable cardinal is weakly inaccessible, so it is a large cardinal. In particular, we obtain results for the ideal of Lebesgue measure zero sets and first Baire category sets.

We will use the above theorem to get analogues of Theorem 1.1.

2. Definitions and notations

The cardinality of a set A is denoted by $|A|$. Cardinal numbers will usually be denoted by κ and λ. The family of all subsets of cardinality not bigger than κ of a set A is denoted by $[A]^{\leq\kappa}$. The set of real numbers is denoted by $\mathbb{R}$. An *ideal* I of subsets of a set X is a family of subsets of X which is closed under finite unions and taking subsets and such that $[X]^{<\omega} \subseteq I$. A family of sets is a *σ-ideal* if it is an ideal and is closed under countable unions.

For a topological space T, by $\mathcal{B}_T$ we denote the family of Borel subsets of T. If I is an ideal of subsets of a set X and $\mathcal{S}$ is a field of subsets of X, then by $\mathcal{S}[I]$ we denote the field generated by $\mathcal{S} \cup I$. If I is a σ-ideal and $\mathcal{S}$ is a σ-field then $\mathcal{S}[I]$ is a σ-field, too.

Let T be an uncountable Polish topological space with σ-finite Borel measure. The σ-ideal of all meagre sets of T is denoted by $\mathbb{K}_T$ and the σ-ideal of measure zero sets is denoted by $\mathbb{L}_T$. The subscript T for $\mathbb{K}_T$ and $\mathbb{L}_T$ will be omitted if T is known from the context. The Lebesgue measure on the real line is denoted by λ. Symbol λ_* denotes the corresponding inner measure. Then $\mathcal{B}_{\mathbb{R}}[\mathbb{L}]$ is the σ-field of Lebesgue measurable subsets of $\mathbb{R}$ and $\mathcal{B}_{\mathbb{R}}[\mathbb{K}]$ is the σ-field of subsets of $\mathbb{R}$ with the Baire property.

If I is an ideal of subsets of a topological space T then we say that the ideal I has a Borel base if for each set $X \in I$ there exists a set $Y \in \mathcal{B}_T$ such that $X \subseteq Y$ and $Y \in I$. The two classical ideals $\mathbb{K}$ and $\mathbb{L}$ have Borel bases.

Let $\mathbb{B}$ be a complete Boolean algebra. We say that $\mathbb{B}$ satisfies c.c.c. (countable chain condition) if every antichain of elements of $\mathbb{B}$ is countable. Boolean algebras $\mathcal{B}_T[\mathbb{K}_T]/\mathbb{K}_T$ and $\mathcal{B}_T[\mathbb{L}_T]/\mathbb{L}_T$ satisfy c.c.c. Assume that $\mathbb{I}$ is a σ-ideal of subsets of Polish space T such that $\mathbb{I}$ has Borel base. We say that $\mathbb{I}$ is c.c.c. if $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ is c.c.c.

We say that the cardinal number κ is *quasi-measurable* if there exists κ-additive ideal $\mathcal{I}$ of subsets of κ such that the Boolean algebra $P(\kappa)/\mathcal{I}$ satisfies c.c.c. Cardinal κ is *weakly inaccessible* if κ is a regular cardinal and for every cardinal $\lambda < \kappa$ we have that $\lambda^+ < \kappa$. Recall that every quasi-measurable cardinal is weakly inaccessible (see [**6**]).

From now on, T denotes an uncountable Polish space. Let us recall the following definition.

Definition 2.1. Let $\mathbb{I}$ be a σ-ideal of subsets of T with Borel base. Let $N \subseteq X \subseteq T$. We say that the set N is *completely* $\mathbb{I}$*-nonmeasurable* in X if

$$(\forall A \in \mathcal{B}_T)(A \cap X \notin \mathbb{I} \to (A \cap N \notin \mathbb{I}) \wedge (A \cap (X \setminus N) \notin \mathbb{I})).$$

In particular, $N \subseteq \mathbb{R}$ is completely $\mathbb{L}$-nonmeasurable if $\lambda_*(N) = 0$ and $\lambda_*(\mathbb{R} \setminus N) = 0$. The definition of completely $\mathbb{K}_T$-nonmeasurability is equivalent to the definition of completely Baire nonmeasurability presented in [**2**] and [**9**].

Complete $\mathbb{I}$-nonmeasurability is also called being $\mathcal{B}_T[\mathbb{I}]$-Bernstein set (see [**2**]).

Definition 2.2. The ideal $\mathbb{I} \subseteq P(T)$ has *the hull property* if for every set $A \subseteq T$ there is an $\mathbb{I}$-minimal Borel set B containing A i.e. if $C \supseteq A$ and C is Borel then $B \setminus C \in \mathbb{I}$. In such case we will write

$$[A]_{\mathbb{I}} = B \quad \text{and} \quad]A[_{\mathbb{I}} = T \setminus [T \setminus A]_{\mathbb{I}}.$$

Let us remark that every c.c.c. σ-ideal with Borel base has the hull property.

Moreover, if the ideal $\mathbb{I}$ has the hull property then $A \subseteq T$ is completely $\mathbb{I}$-nonmeasurable if and only if $[A]_{\mathbb{I}} = T$ and $]A[_{\mathbb{I}} = \emptyset$.

3. Baire-nonmeasurable sets

We will need the following theorem from [**5**]:

THEOREM 3.1 (Gitik, Shelah). *If I is a σ-ideal on κ, then $P(\kappa)/I$ is not isomorphic to the Cohen algebra.*

In this section we will use the following notation. For $A, B \subseteq T$ let

$$A \subseteq^* B \quad \text{iff} \quad B \setminus A \in \mathbb{K}.$$

THEOREM 3.2. *Assume that $\mathcal{A} \subseteq \mathbb{K}$ is a partition of a subset of T and $\bigcup \mathcal{A} \notin \mathbb{K}$. Let $\mathcal{A}_n \subseteq \mathcal{A}$ for $n \in \omega$. Then there exists $\mathcal{B} \subseteq \mathcal{A}$ such that*

(1) $\bigcup \mathcal{B} \notin \mathbb{K}$,
(2) *for every $n \in \omega$, $\bigcup \mathcal{B} \cap \bigcup \mathcal{A}_n$ does not contain any $\mathbb{K}$-possitive Borel set modulo $\bigcup \mathcal{A}_n$ i.e.*

$$(\forall n)(\neg \exists U)(U \textit{is open} \wedge U \cap \bigcup \mathcal{A}_n \notin \mathbb{K} \wedge U \cap \bigcup \mathcal{A}_n \subseteq^* \bigcup \mathcal{B}).$$

PROOF. Assume to the contrary that some partition $\mathcal{A} \subseteq \mathbb{K}$ of a subset of T such that $\bigcup \mathcal{A} \notin \mathbb{K}$ and some $\mathcal{A}_n \subseteq \mathcal{A}$ for $n \in \omega$ satisfy the following condition for every $\mathcal{B} \subseteq \mathcal{A}$

$$\bigcup \mathcal{B} \notin \mathbb{K} \to (\exists n \in \omega)(\exists U)(U \text{is open} \wedge \mathbb{K} \not\ni U \cap \bigcup \mathcal{A}_n \subseteq^* \bigcup \mathcal{B}).$$

Fix a base $\mathcal{U} = \{U_n : n \in \omega\}$. For $n, k \in \omega$ such that $U_n \cap \bigcup \mathcal{A}_k \notin \mathbb{K}$ consider a family

$$\mathcal{D}_{n,k} = \{\mathcal{X} \subseteq \mathcal{A} : U_n \cap \bigcup \mathcal{A}_k \subseteq^* \bigcup \mathcal{X}\}$$

Let $\mathcal{K} = \{\mathcal{X} \subseteq \mathcal{A} : \bigcup \mathcal{X} \in \mathbb{K}\}$. By our assumption $P(\mathcal{A}) = \bigcup_{n,k \in \omega} \mathcal{D}_{n,k} \cup \mathcal{K}$. Algebra $P(\mathcal{A})/\mathcal{K}$ is clearly c.c.c. and complete. So, we can find

$$C_{n,k} = \prod \mathcal{D}_{n,k} = \bigcap \{X_m : m \in \omega\} \neq \emptyset$$

for some $\{X_m : m \in \omega\} \subseteq \mathcal{D}_{n,k}$. The family $\{C_{n,k} : n, k \in \omega\}$ is dense in algebra $P(\mathcal{A})/\mathcal{K}$. What is more $P(\mathcal{A})/\mathcal{K}$ is atomless, since $|\mathcal{A}| \leq 2^\omega$ witness that $|\mathcal{A}|$ is not a measurable cardinal. Summarising, we can see that $P(\mathcal{A})/\mathcal{K}$ is atomless algebra with countable dense subset, so it is izomorphic to the Cohen algebra contradicting Theorem 3.1. $\square$

DEFINITION 3.3. Let κ, λ, μ, ν be cardinal numbers. The relation $(\kappa\colon \lambda, \mu) \to \nu$ holds if for every family $\mathcal{R}$ of μ-additive ideals on κ such that $|\mathcal{R}| = \lambda$ there exists a family $\{X_\alpha\}_{\alpha<\nu}$ such that

(1) $(\forall\alpha<\nu)(X_\alpha\in P(\kappa)\setminus\bigcup\mathcal{R})$,
(2) $(\forall\alpha<\beta<\nu)(X_\alpha\cap X_\beta\in\bigcap\mathcal{R})$.

THEOREM 3.4 (Alaoglu, Erdös). *For every cardinal κ*

$$((\kappa\colon\omega,\omega_1)\to\omega_1)\leftrightarrow((\kappa\colon 1,\omega_1)\to\omega_1).$$

We will need the following version of Theorem 3.4, which follows easily from the proof of Theorem 3.4 from [**8**]:

LEMMA 3.5. *Assume that $\{I_n\}_{n\in\omega}$ is a family of σ-additive ideals on κ which are not c.c.c. Then there exists a family $\{X_\alpha\}_{\alpha<\omega_1}\subseteq P(\kappa)$ such that*

(1) $(\forall\alpha<\omega_1)(\forall n\in\omega)(X_\alpha\notin I_n)$
(2) $(\forall\alpha,\beta<\omega_1)(\alpha\neq\beta\to X_\alpha\cap X_\beta=\emptyset)$.

THEOREM 3.6. *Assume that $\mathcal{A}\subseteq\mathbb{K}$ is a partition of a subset of T. Let $\mathcal{A}_n\subseteq\mathcal{A}$ for $n\in\omega$. Then there exists $\mathcal{B}\subseteq\mathcal{A}$ such that*

(1) $[\bigcup\mathcal{B}]_{\mathbb{K}}=[\bigcup\mathcal{A}]_{\mathbb{K}}$,
(2) *for every $n\in\omega$, $\bigcup\mathcal{B}\cap\bigcup\mathcal{A}_n$ does not contain any $\mathbb{K}$-possitive Borel set modulo $\bigcup\mathcal{A}_n$ i.e.*

$$(\forall n)(\neg\exists U)(U \textit{ is open}\wedge U\cap\bigcup\mathcal{A}_n\notin\mathbb{K}\wedge U\cap\bigcup\mathcal{A}_n\subseteq^*\bigcup\mathcal{B}).$$

PROOF. Let $\bigcup\mathcal{A}=Z\notin\mathbb{K}$. Fix a base $\{U_n : n\in\omega\}$ of T. Define

$$Z_1=\bigcup\{U_n\cap\bigcup\mathcal{A}_k\ :\ U_n\cap\bigcup\mathcal{A}_k\notin\mathbb{K}\wedge P(\mathcal{A})/\mathcal{K}_{n,k}\text{ is c.c.c. }\},$$

where

$$\mathcal{K}_{n,k}=\{\mathcal{X}\subseteq\mathcal{A}:\bigcup\mathcal{X}\cap\bigcup\mathcal{A}_k\cap U_n\in\mathbb{K}\}.$$

Let $Z_0=Z\setminus Z_1$ and

$$\mathcal{K}^1=\{\mathcal{X}\subseteq\mathcal{A}:\bigcup\mathcal{X}\cap Z_1\in\mathbb{K}\}.$$

Then for every $n,k\in\omega$

$$\text{if } Z_0\cap U_n\cap\bigcup\mathcal{A}_k\notin\mathbb{K}\text{ then } P(\mathcal{A})/\mathcal{K}_{n,k}\text{ is not c.c.c.}$$

and

$$P(\mathcal{A})/\mathcal{K}^1\text{ is c.c.c.}$$

First, work with Z_0 part. By Lemma 3.5 we can find a family $\{\mathcal{B}^0_\xi:\xi\in\omega_1\}\subseteq P(\mathcal{A})$ of pairwise disjoint sets such that

$$\mathcal{B}^0_\xi\notin\mathcal{K}_{n,k}\text{ for every } n,k \text{ such that } Z_0\cap U_n\cap\bigcup\mathcal{A}_k\notin\mathbb{K}$$

Now by c.c.c.-ness of Z_1 part we can find two ordinals ξ_0,ξ_1 such that

$$\bigcup\mathcal{B}^0_{\xi_i}\cap Z_1\in\mathbb{K}.$$

Now work with the family $\mathcal{A}\setminus(\mathcal{B}^0_{\xi_0}\cup\mathcal{B}_{\xi_1})$ By Theorem 3.2 we can find $\mathcal{B}^1_0\subseteq\mathcal{A}\setminus(\mathcal{B}^0_{\xi_0}\cup\mathcal{B}_{\xi_1})$ such that

(1) $\bigcup\mathcal{B}^1_0\notin\mathbb{K}$,
(2) for every $n\in\omega$, $\bigcup\mathcal{B}^1_0\cap\bigcup\mathcal{A}_n$ does not contain any $\mathbb{K}$-possitive Borel set modulo $\bigcup\mathcal{A}_n$

We can proceed by induction finding $\mathcal{B}^1_\alpha\subseteq\mathcal{A}\setminus(\mathcal{B}^0_{\xi_0}\cup\mathcal{B}^0_{\xi_1}\cup\bigcup_{\beta<\alpha}\mathcal{B}^1_\beta)$

(1) $\bigcup\mathcal{B}^1_\alpha\notin\mathbb{K}$,
(2) for every $n\in\omega$, $\bigcup\mathcal{B}^1_\alpha\cap\bigcup\mathcal{A}_n$ does not contain any $\mathbb{K}$-possitive Borel set modulo $\bigcup\mathcal{A}_n$

if $\bigcup\left(\mathcal{A}\setminus(\mathcal{B}^0_{\xi_0}\cup\mathcal{B}^0_{\xi_1}\cup\bigcup_{\beta<\alpha}\mathcal{B}^1_\beta)\right)\cap Z_1\notin\mathbb{K}$. We are in Z_1 part, so our construction ends after $\lambda<\omega_1$ steps. Now by simple induction on $(n,k)\in\omega\times\omega$ we can find $\mathcal{B}_{n,k}=\mathcal{B}^1_\xi,\mathcal{B}'_{n,k}=\mathcal{B}^1_\zeta$ for some $\xi,\zeta<\lambda$ such that

$$\bigcup\mathcal{B}_{n,k}\cap U_n\cap\bigcup\mathcal{A}_k\notin\mathbb{K}\quad\text{and}\quad\bigcup\mathcal{B}'_{n,k}\cap U_n\cap\bigcup\mathcal{A}_k\notin\mathbb{K}$$

if $Z_1\cap U_n\cap\bigcup\mathcal{A}_k\notin\mathbb{K}$. Now put $\mathcal{B}=\mathcal{B}^0_{\xi_0}\cup\bigcup_{n,k\in\omega}\mathcal{B}_{n,k}$. This set $\mathcal{B}$ fulfills our conditions. □

THEOREM 3.7. *Assume that $\mathcal{A}\subseteq\mathbb{K}$ is a partition of T. Let $\mathcal{A}_n\subseteq\mathcal{A}$ for $n\in\omega$. Then there exists $\mathcal{B}_n\subseteq\mathcal{A}_n$ such that*

(1) $\mathcal{B}_n\cap\mathcal{B}_m=\emptyset$ *for* $n\neq m$,
(2) $\mathcal{B}_n\subseteq\mathcal{A}_n$,
(3) $[\bigcup\mathcal{B}_n]_\mathbb{K}=[\bigcup\mathcal{A}_n]_\mathbb{K}$.

PROOF. Using Theorem 3.6 (for $\mathcal{A}'=\mathcal{A}_0$ and $\mathcal{A}'_n=\mathcal{A}_n\cap\mathcal{A}_0$) we can find $\mathcal{B}_0\subseteq\mathcal{A}_0$ such that

(1) $[\bigcup\mathcal{B}_0]_\mathbb{K}=[\bigcup\mathcal{A}_0]_\mathbb{K}$,
(2) $]\bigcup\mathcal{B}_0\cap\bigcup\mathcal{A}_n[_\mathbb{K}=\emptyset$ for every $n\in\omega$.

It means that we can proceed by simple induction considering at k-th step a family $\{\mathcal{A}^k_n : n\in\omega\}$, where $\mathcal{A}^k_n=\mathcal{A}_n\setminus(\mathcal{B}_0\cup\ldots\cup\mathcal{B}_{k-1})$. We have that $[\bigcup\mathcal{A}_n]_\mathbb{K}=[\bigcup\mathcal{A}^k_n]_\mathbb{K}$. So it is enough to work with family $\{\mathcal{A}^k_n : n\geq k\}$ and find a suitable subset $\mathcal{B}_k\subseteq\mathcal{A}^k_k$. □

4. $\mathbb{I}$-nonmeasurable sets

From now on, $\mathbb{I}\subseteq P(T)$ denotes c.c.c. σ-ideal with Borel base. We will use the following result (see [**9**], [**7**]).

Theorem 4.1 (Rałowski, Żeberski). *Assume that there is no quasi-measurable cardinal smaller than* 2^ω. *Let* $\mathcal{A} \subseteq \mathbb{I}$ *be a point-finite family such that* $\bigcup \mathcal{A} \notin \mathbb{I}$. *Then there exists a collection of pairwise disjoint subfamilies* $\mathcal{A}_\xi \subseteq \mathcal{A}$ *(for* $\xi \in \omega_1$*) such that* $\bigcup \mathcal{A}_\xi$ *is completely* $\mathbb{I}$*-nonmeasurable in* $\bigcup \mathcal{A}$.

We will also need the following lemma (see [**9**]).

Lemma 4.2 (Żeberski). *Let* $\mathcal{A} \subseteq P(T)$ *be any point-finite family. Then there exists a subfamily* $\mathcal{A}' \subseteq \mathcal{A}$ *such that* $|\mathcal{A} \setminus \mathcal{A}'| \leq \omega$ *and*

$$(\forall B \in \mathcal{B}_T[\mathbb{I}])(\forall A \in \mathcal{A}')(B \cap \bigcup \mathcal{A} \notin \mathbb{I} \to \neg(B \cap \bigcup \mathcal{A} \subseteq B \cap A)).$$

Lemma 4.3. *Assume that there is no quasi-measurable cardinal smaller than continuum. Assume that* $\mathcal{A} \subseteq \mathbb{I}$ *is point-finite family. Let* $(\mathcal{A}_n :\ n \in \omega)$ *be a sequence of subsets of* $\mathcal{A}$. *Then we can find a sequence* $(\mathcal{B}_n^: n \in \omega)$ *such that*

(1) $\mathcal{B}_n \cap \mathcal{B}_m = \emptyset$ *for* $n \neq m$,
(2) $\mathcal{B}_n \subseteq \mathcal{A}_n$,
(3) $[\bigcup \mathcal{A}_n]_\mathbb{I} = [\bigcup \mathcal{B}_n]_\mathbb{I}$.

Proof. Let us put $\mathcal{A}_n^0 = \mathcal{A}_n$ for every $n \in \omega$. We will proceed by induction. At k-th step for each $n \geq k$ we have a subset $\mathcal{A}_n^k \subseteq \mathcal{A}_n$ such that $[\bigcup \mathcal{A}_n^k]_\mathbb{I} = [\bigcup \mathcal{A}_n]_\mathbb{I}$. Using Theorem 4.1 we can find $(\mathcal{B}_k^\alpha)_{\alpha\in\omega_1}$ such that $\bigcup \mathcal{B}_k^\alpha$ is completely $\mathbb{I}$-nonmeasurable in $\bigcup \mathcal{A}_k^k$. Now, by Lemma 4.2 there are at most countably many α's such that $[\bigcup \mathcal{A}_n^k \setminus \bigcup \mathcal{B}_k^\alpha]_\mathbb{I} \neq [\bigcup \mathcal{A}_n^k]_\mathbb{I}$ for some $n \in \omega$. So, we can find $\mathcal{B}_k$ such that

(1) $\mathcal{B}_k \subseteq \mathcal{A}_k^k$,
(2) $[\bigcup \mathcal{B}_k]_\mathbb{I} = [\bigcup \mathcal{A}_k^k]_\mathbb{I} = [\bigcup \mathcal{A}_k]_\mathbb{I}$,
(3) $[\bigcup \mathcal{A}_n^k \setminus \bigcup \mathcal{B}_k]_\mathbb{I} = [\bigcup \mathcal{A}_n^k]_\mathbb{I}$ for every $n \geq k$.

So, for each $n > k$ we can put $\mathcal{A}_n^{k+1} = \mathcal{A}_n^k \setminus \mathcal{B}_k$ and continue our procedure. □

Theorem 4.4. *Assume that there is no quasi-measurable cardinal smaller than continuum. Assume that* $\mathcal{A} \subseteq \mathbb{I}$ *is point-finite family. Let* $(\mathcal{A}_n :\ n \in \omega)$ *be a sequence of subsets of* $\mathcal{A}$. *Then we can find a sequence* $(\mathcal{B}_n^\xi :\ n \in \omega,\ \xi \in \omega_1)$ *such that*

(1) $\mathcal{B}_n^\xi \cap \mathcal{B}_m^\zeta = \emptyset$ *for* $(n,\xi) \neq (m,\zeta)$,
(2) $\mathcal{B}_n^\xi \subseteq \mathcal{A}_n$,
(3) $[\bigcup \mathcal{A}_n]_\mathbb{I} = [\bigcup \mathcal{B}_n^\xi]_\mathbb{I}$.

Proof. First let us use Lemma 4.3 to obtain a collection $\{\mathcal{B}_n :\ n \in \omega\}$ such that

(1) $\mathcal{B}_n \cap \mathcal{B}_m = \emptyset$ for $n \neq m$,

(2) $\mathcal{B}_n \subseteq \mathcal{A}_n$,
(3) $[\bigcup \mathcal{A}_n]_{\mathbb{I}} = [\bigcup \mathcal{B}_n]_{\mathbb{I}}$.

Now using Theorem 4.1 for each $n \in \omega$ we can find $(\mathcal{B}_n^\alpha : \ \alpha \in \omega_1)$ such that

(1) $\mathcal{B}_n^\alpha \subseteq \mathcal{B}_n$,
(2) $\mathcal{B}_n^\alpha \cap \mathcal{B}_n^\beta = \emptyset$ for $\alpha \neq \beta$,
(3) $\bigcup \mathcal{B}_n^\alpha$ is completely $\mathbb{I}$-nonmeasrable in $\bigcup \mathcal{B}_n$.

The collection $(\mathcal{B}_n^\alpha : \ n \in \omega, \alpha \in \omega_1)$ fulfills desired conditions. $\square$

5. Acknowledgements

The author would like to thank the referee for careful revision which gives improvement of the presentation of the paper.

References

[1] J. Brzuchowski, J. Cichoń, E. Grzegorek, C. Ryll-Nardzewski, *On the existence of nonmeasurable unions*, Bull. Polish Acad. Sci. Math. **27** (1979), 447-448,

[2] J. Cichoń, M. Morayne, R. Rałowski, C. Ryll-Nardzewski, S. Żeberski, *On nonmeasurable unions,* Topology and its Applications 154 (2007), 884-893,

[3] P. Erdös, *Some remarks on set theory,* Proc. Amer. Math. Soc. **1** (1950), 127-141,

[4] M. Gitik, S. Shelah, *More on real-valued measurable cardinals and forcings with ideals,* Israel Journal of Mathematics 124 (1), (2001), 221-242,

[5] M. Gitik, S. Shelah, *Forcing with ideals and simple forcing notions,* Israel J. Math. **62** (1989), 129-160,

[6] T. Jech, *Set Theory* Springer-Verlag, 2000,

[7] R. Rałowski, S. Żeberski, *Completely nonmeasurable union*, preprint,

[8] A. Taylor, *On saturated sets of ideals and Ulam's problem,* Fund. Math. **109** (1980), 37-53,

[9] S. Żeberski, *On completely nonmeasurable unions*, Mathematical Logic Quarterly 53 (1) (2007), 38-42.

Institute of Mathematics and Computer Science, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland.

E-mail address: szymon.zeberski@pwr.wroc.pl