GENERALIZED LUZIN SETS

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Abstract. In this paper we investigate the notion of a generalized $(\mathcal{I}, \mathcal{J})$ - Luzin set. This notion generalizes the standard notion of a Luzin set and a Sierpiński set. We find set theoretical conditions which imply the existence of a generalized $(\mathcal{I}, \mathcal{J})$ - Luzin set. We show how to construct a large family of pairwise non-equivalent $(\mathcal{I}, \mathcal{J})$ - Luzin sets. We find a class of forcings which preserves the property of being a $(\mathcal{I}, \mathcal{J})$ - Luzin set.

1. Notation and Terminology

We will use standard set-theoretic notation following [10]. In particular for any set X and any cardinal κ, $[X]^{<\kappa}$ denotes the set of all subsets of X with size less than κ. Similarly, $[X]^{\kappa}$ denotes the family of subsets of X of size κ. By $\mathscr{P}(X)$ we denote the power set of X.

If $A \subseteq X \times Y$ then for $x \in X$ and $y \in Y$ we put

$$A_x = \{y \in Y : (x, y) \in A\},$$

$$A^y = \{x \in X : (x, y) \in A\}.$$

By $A \triangle B$ we denote the symmetric difference of sets A and B, i.e.

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

In this paper $\mathscr{X}$ denotes an uncountable Polish space. By $\mathrm{Open}(\mathscr{X})$ we denote the topology of $\mathscr{X}$. By $\mathrm{Borel}(\mathscr{X})$ we denote the σ-field of all Borel sets. Let us recall that each Borel set can be coded by a function from ω^ω. A precise definition of such coding can be found in [9]. If $x \in \omega^\omega$ is a Borel code then by $\#x$ we denote the Borel set coded by x.

$\mathcal{I}$, $\mathcal{J}$ are σ-ideals on $\mathscr{X}$, i.e. $\mathcal{I}, \mathcal{J} \subseteq \mathscr{P}(\mathscr{X})$ are closed under countable unions and subsets. Additionally we assume that $[\mathscr{X}]^\omega \subseteq \mathcal{I}, \mathcal{J}$. Moreover we assume that $\mathcal{I}, \mathcal{J}$ have a Borel base i.e each set from the ideal can be covered by a Borel set from the ideal. Standard examples of such ideals are the ideal $\mathbb{L}$ of Lebesgue measure zero sets and the ideal $\mathbb{K}$ of meager sets of Polish space.

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Definition 1.1. *Let $M \subseteq N$ be standard transitive models of ZF. Coding Borel sets from the ideal I is absolute iff*

$$(\forall x \in M \cap \omega^\omega)(M \models \# x \in I \longleftrightarrow N \models \# x \in I).$$

We say that $\mathcal{I}$ is κ saturated (κ-c.c.) if every family $\mathscr{A}$ of Borel subsets of $\mathscr{X}$ satisfying the following conditions:

(1) $(\forall A \in \mathscr{A})(A \notin \mathcal{I})$
(2) $(\forall A, B \in \mathscr{A})(A \neq B \to A \cap B \in \mathcal{I})$

has size smaller than κ. If $\mathcal{I}$ is ω_1-c.c. then we say that $\mathcal{I}$ is c.c.c.

Let us recall that a function $f : \mathscr{X} \to \mathscr{X}$ is $\mathcal{I}$-measurable if the preimage of every open subset of $\mathscr{X}$ is $\mathcal{I}$-measurable i.e belongs to the σ-field generated by Borel sets and the ideal $\mathcal{I}$. In other words f is $\mathcal{I}$-measurable iff

$$(\forall U \in \mathrm{Open}(\mathscr{X}))(\exists B \in \mathrm{Borel}(\mathscr{X}))(\exists I \in \mathcal{I})(f^{-1}[U] = B \triangle I).$$

Let us recall the following cardinal coefficients:

Definition 1.2 (Cardinal coefficients)**.**

$$\begin{aligned}
non(\mathcal{I}) &= \min\{|A| : A \subseteq \mathscr{X} \wedge A \notin \mathcal{I}\} \\
add(\mathcal{I}) &= \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathcal{I} \wedge \bigcup \mathscr{A} \notin \mathcal{I}\} \\
cov(\mathcal{I}) &= \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathcal{I} \wedge \bigcup \mathscr{A} = \mathscr{X}\} \\
cov_h(\mathcal{I}) &= \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathcal{I} \wedge (\exists B \in \mathrm{Borel}(\mathscr{X}) \setminus \mathcal{I})(B \subseteq \bigcup \mathscr{A})\} \\
cof(\mathcal{I}) &= \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathcal{I} \wedge \mathscr{A} \textit{ is a base of } \mathcal{I}\}
\end{aligned}$$

where $\mathscr{A}$ is a base of $\mathcal{I}$ iff $\mathscr{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathscr{A})(I \subseteq A)$.

Let us remark that the above coefficients can be defined for a larger class of families (not only ideals).

Definition 1.3. *We say that $L \subseteq \mathscr{X}$ is a $(\mathcal{I}, \mathcal{J})$ - Luzin set if*

- $L \notin \mathcal{I}$,
- $(\forall B \in \mathcal{I})(B \cap L \in \mathcal{J})$.

Assume that κ is a cardinal number. We say that $L \subseteq \mathscr{X}$ is a $(\kappa, \mathcal{I}, \mathcal{J})$ - Luzin set iff L is a $(\mathcal{I}, \mathcal{J})$ - Luzin set and $|L| = \kappa$.

The above definition generalizes the standard notion of Luzin and Sierpiński sets. Namely, L is a Luzin set iff L is a generalized $(\mathbb{L}, [\mathbb{R}]^{\leq\omega})$ - Luzin set and S is a Sierpiński set iff S is a generalized $(\mathbb{K}, [\mathbb{R}]^{\leq\omega})$ - Luzin set. The above notion generalizes also notions from [3, 1].

Definition 1.4. *We say that ideals $\mathcal{I}$ and $\mathcal{J}$ are orthogonal if*

$$\exists A \in \mathscr{P}(\mathscr{X})\ A \in I \wedge A^c \in \mathcal{J}.$$

In this case we write $\mathcal{I} \perp \mathcal{J}$.

Definition 1.5. *Let $\mathscr{F} \subseteq \mathscr{X}^{\mathscr{X}}$ be a family of functions. We say that $A, B \subseteq \mathscr{X}$ are equivalent with respect to $\mathscr{F}$ if*

$$(\exists f \in \mathscr{F})\ (B = f[A] \vee A = f[B])$$

Definition 1.6. *We say that* $A, B \subseteq \mathscr{X}$ *are Borel equivalent if* A, B *are equivalent with respect to the family of all Borel functions.*

Definition 1.7. *We say that* $\mathcal{I}$ *has the Fubini property iff for every Borel set* $A \subseteq \mathscr{X} \times \mathscr{X}$

$$\{x \in \mathscr{X} : A_x \notin \mathcal{I}\} \in \mathcal{I} \Longrightarrow \{y \in \mathscr{X} : A^y \notin \mathcal{I}\} \in \mathcal{I}$$

Natural examples of ideals fulfilling the Fubini property are the ideal $\mathbb{L}$ of null sets (by the Fubini theorem) and the ideal $\mathbb{K}$ of meager sets (by the Kuratowski-Ulam theorem).

From the definition we can obtain the following properties:

Fact 1.1. *Assume that* $\mathcal{I} \perp \mathcal{J}$.

(1) *There exist a* $(\mathcal{I}, \mathcal{J})$ *- Luzin set.*
(2) *If* L *is a* $(\mathcal{I}, \mathcal{J})$ *- Luzin set then* L *is not a* $(\mathcal{J}, \mathcal{I})$ *- Luzin set.*

Proof. (Part 1) By the definition of $\mathcal{I} \perp \mathcal{J}$ we can find two sets $I \in \mathcal{I}$ and $J \in \mathcal{J}$ such that $I \cup J = \mathscr{X}$. We will show that J is a $(\mathcal{I}, \mathcal{J})$ - Luzin set. J is not in $\mathcal{I}$. Let us fix any set $A \in \mathcal{I}$. We have that $A \cap J \subseteq J \in \mathcal{J}$.

(Part 2) By the definition of $\mathcal{I} \perp \mathcal{J}$ we can find two sets $I \in \mathcal{I}$ and $J \in \mathcal{J}$ such that $I \cup J = \mathscr{X}$. Assume that L is a $(\mathcal{I}, \mathcal{J})$ - Luzin set and a $(\mathcal{J}, \mathcal{I})$ - Luzin set. We have that

$$L \cap J \subseteq J \in \mathcal{J} \text{ and } L \cap I \subset I \in \mathcal{I}$$

By the property of being a $(\mathcal{J}, \mathcal{I})$ - Luzin set

$$L \cap J \in \mathcal{I}.$$

So $L = (L \cap J) \cup (L \cap I) \in \mathcal{I}$ what is a contradiction with being a $(\mathcal{I}, \mathcal{J})$ - Luzin set. □

We will try to find a wide class of forcings which preserves the property of being a $(\mathcal{I}, \mathcal{J})$ - Luzin set. We are mainly interested in so called definable forcings (see [14]). Let us recall that $\mathbb{P}$ is a definable forcing if $\mathbb{P}$ is of the form $\text{Borel}(\mathscr{X}) \setminus \mathcal{I}$, where $\mathscr{X}$ and $\mathcal{I}$ have absolute definition for standard transitive models of ZF of the same height.

2. Existence of Luzin sets

Let us start with a theorem which under suitable assumptions guarantees existence of uncountably many pairwise different $(\mathcal{I}, \mathcal{J})$ - Luzin sets. This result generalize the Erdös-Sierpinski Duality Theorem (see Theorem 19.6 in [5]).

Theorem 2.1. *Assume that* $\kappa = cov(\mathcal{I}) = cof(\mathcal{I}) \leq non(\mathcal{J})$. *Let* $\mathscr{F}$ *be a family of functions from* $\mathscr{X}$ *to* $\mathscr{X}$. *Assume that* $|\mathscr{F}| \leq \kappa$. *Then we can find a sequence* $(L_\alpha)_{\alpha<\kappa}$ *such that*

(1) L_α *is a* $(\kappa, \mathcal{I}, \mathcal{J})$ *- Luzin set,*

(2) *for* $\alpha \neq \beta$, L_α *is not equivalent to* L_β *with respect to the family* $\mathscr{F}$.

Proof. Let us enumerate the family $\mathscr{F}$:

$$\mathscr{F} = \{f_\alpha : \alpha < \kappa\}.$$

Now, let us enumerate Borel base of ideal $\mathcal{I}$:

$$\mathscr{B}_\mathcal{I} = \{B_\alpha :\ \alpha < \kappa\}.$$

Now without loss of generality we can assume that

$$(\forall f \in \mathscr{F})(\forall \lambda < \kappa)(\kappa \leq |f[(\bigcup_{\xi<\lambda} B_\xi)^c]|).$$

Indeed, since $cov(\mathcal{I}) = \kappa$ a set $(\bigcup_{\xi<\lambda} B_\xi)^c$ is not in the ideal $\mathcal{I}$. If the function f does not have the above property and L is a $(\mathcal{I}, \mathcal{J})$-Luzin set then

$$f[L] = f[L \cap \bigcup_{\xi<\lambda} B_\xi] \cup f[L \cap (\bigcup_{\xi<\lambda} B_\xi)^c]$$

and both sets have cardinality less than κ. So $f[L]$ is not a $(\mathcal{I}, \mathcal{J})$-Luzin set.

By induction we will construct the family $\{x^\eta_{\alpha,\zeta} : \eta, \zeta, \alpha < \kappa\}$ and $\{d^\eta_{\alpha,\zeta} : \eta, \zeta, \alpha < \kappa\}$ such that

$$d^\eta_{\alpha,\zeta} = f_\zeta(x^\eta_{\alpha,\zeta})$$

and for any different $\eta, \eta' < \kappa$

$$\{x^\eta_{\alpha,\zeta} : \zeta, \alpha < \kappa\} \cap \{d^{\eta'}_{\alpha,\zeta} : \zeta, \alpha < \kappa\} = \emptyset$$

and

$$x^\eta_{\alpha,\zeta} \in \mathscr{X} \setminus \left(\{d^\eta_{\xi,\zeta} :\ \eta, \xi, \zeta < \alpha\} \cup \{x^\eta_{\xi,\zeta} :\ \eta, \xi, \zeta < \alpha\} \cup \bigcup_{\xi<\alpha} B_\xi\right)$$

for every $\eta, \zeta < \alpha$.

Assume that we are in the α-th step of construction. Fix $\eta, \zeta < \alpha$. It means that we have constructed the following set

$$Old = \{x^\lambda_{\beta,\xi}, d^\lambda_{\beta,\xi} :\ \beta, \xi, \lambda < \alpha\} \cup \{x^\lambda_{\alpha,\xi}, d^\lambda_{\alpha,\xi} :\ \lambda < \eta \vee (\lambda = \eta \wedge \xi < \zeta)\}.$$

Since $|f_\zeta[(\bigcup_{\xi<\alpha} B_\xi)^c]| \geq \kappa$ and $|Old| < \kappa$ we get that

$$|f_\zeta[(\bigcup_{\xi<\alpha} B_\xi \cup Old)^c]| \geq \kappa.$$

That's why we can find

$$d^\eta_{\alpha,\zeta} \in f_\zeta[(\bigcup_{\xi<\alpha} B_\xi \cup Old)^c] \setminus Old.$$

Let $x^\eta_{\alpha,\zeta}$ be such that $d^\eta_{\alpha,\zeta} = f_\zeta(x^\eta_{\alpha,\zeta})$. In this way we can finish the α-th step of construction.

Now, let us define $L_\alpha = \{x^\alpha_{\xi,\zeta} : \xi, \zeta < \kappa\}$.

Let us check that L_α is a $(\mathcal{I},\mathcal{J})$ - Luzin set. Indeed, if $A\in\mathcal{I}$ then there exists $\beta<\kappa$ s.t. $A\subset B_\beta$. Then we have

$$A\cap L_\alpha\subset B_\beta\cap L_\alpha=B_\beta\cap\{x^\alpha_{\xi,\zeta}:\ \xi,\zeta<\beta\}\subseteq\{x^\alpha_{\xi,\zeta}:\ \xi,\zeta<\beta\}\in\mathcal{J}$$

because $|\{x^\alpha_{\xi,\zeta}:\ \xi,\zeta<\beta\}|\leq|\beta|<\kappa\leq non(\mathcal{J})$.

What is more, for every function $f=f_\alpha\in\mathscr{F}$ and every $\beta\neq\gamma$ we have that

$$\kappa\leq|f[L_\gamma]\setminus L_\beta|$$

because $\{d^\gamma_{\xi,\alpha}:\alpha<\xi<\kappa\}\subseteq f[L_\gamma]\setminus L_\beta$. So $L_\beta\neq f[L_\gamma]$. □

In fact we haved proved a little stronger result.

Remark 2.1. *Assume that $\kappa=cov(\mathcal{I})=cof(\mathcal{I})\leq non(\mathcal{J})$. Let $\mathscr{F}$ be a family of functions from $\mathscr{X}$ to $\mathscr{X}$. Assume that $|\mathscr{F}|\leq\kappa$. Then we can find a sequence $(L_\alpha)_{\alpha<\kappa}$ such that*

1. *L_α is $(\kappa,\mathcal{I},\mathcal{J})$ - Luzin set,*
2. *for $\alpha\neq\beta$ and $f\in\mathscr{F}$ we have that $\kappa\leq|f[L_\alpha]\triangle L_\beta|$.*

Let us notice that for every ideal $\mathcal{I}$ we have the inequality $cov(\mathcal{I})\leq cof(\mathcal{I})$. This gives the following corollary.

Corollary 2.1. *If $2^\omega=cov(\mathcal{I})=non(\mathcal{J})$ then there exists continuum many different $(\mathcal{I},\mathcal{J})$ - Luzin sets which aren't Borel equivalent.*

In particular, if CH holds then there exists continuum many different $(\omega_1,\mathcal{I},\mathcal{J})$ - Luzin sets which aren't Borel equivalent.

We can extend above corollary to a wilder class of functions - namely, $\mathcal{I}$-measurable functions.

Corollary 2.2. *If $2^\omega=cov(\mathcal{I})=non(\mathcal{J})$ then there exist continuum many different $(\mathcal{I},\mathcal{J})$ - Luzin sets which aren't equivalent with respect to all $\mathcal{I}$-measurable functions.*

In particular, if CH holds then there exists continuum many different $(\omega_1,\mathcal{I},\mathcal{J})$ - Luzin sets which aren't equivalent with respect to all $\mathcal{I}$-measurable functions.

Proof. First, let us notice that if a function f is $\mathcal{I}$-measurable then there exists a set $I\in\mathcal{I}\cap\text{Borel}(\mathscr{X})$ such that $f\upharpoonright(\mathscr{X}\setminus I)$ is Borel. Indeed, it is enough to consider a countable base $\{U_n\}_{n\in\omega}$ of topology of $\mathscr{X}$. Then $f^{-1}[U_n]=B_n\triangle I_n$, where B_n is Borel and I_n is from the ideal $\mathcal{I}$. Now, put $I=\bigcup_{n\in\omega}I_n$.

So we can consider a family of partial Borel functions whose domain is Borel set with complement in the ideal $\mathcal{I}$. This family is naturally of size continuum. So we can use Corollary 2.1 and Remark 2.1 to finish the proof. □

Now, let us concentrate on the ideals of null and meager sets.

Corollary 2.3. (1) *Assume that $cov(\mathbb{L}) = 2^\omega$. There exists continuum many different $(2^\omega, \mathbb{L}, \mathbb{K})$ - Luzin sets which aren't equivalent with respect to the family of Lebesgue - measurable functions.*
(2) *Assume that $cov(\mathbb{K}) = 2^\omega$. There exists continuum many different $(2^\omega, \mathbb{K}, \mathbb{L})$ - Luzin sets which aren't equivalent with respect to the family of Baire - measurable functions.*

Proof. Let us notice that the equality $cov(\mathbb{L}) = 2^\omega$ implies that $2^\omega = cov(\mathbb{L}) = cof(\mathbb{L}) = non(\mathbb{K})$. Similarly, the equality $cov(\mathbb{K}) = 2^\omega$ implies that $2^\omega = cov(\mathbb{K}) = cof(\mathbb{K}) = non(\mathbb{L})$ (see [2]). Corollary 2.2 finishes the proof. □

3. Luzin sets and forcing

Now, let us focus on the class of forcings which preserve being a $(\mathcal{I}, \mathcal{J})$-Luzin set. Let us start with a technical observation.

Lemma 3.1. *Assume that $\mathcal{I}$ has the Fubini property. Suppose that $\mathbb{P}_\mathcal{I} = \mathrm{Borel}(\mathscr{X}) \setminus \mathcal{I}$ is a proper definable forcing. Let $B \in \mathcal{I}$ be a set in $V^{\mathbb{P}_\mathcal{I}}[G]$. Then there exists a set $D \in \mathcal{I} \cap V$ such that $B \cap \mathscr{X}^V \subseteq D$.*

Proof. Because the ideal $\mathcal{I}$ has Borel base then let $\dot{B}$ denote the $\mathbb{P}_\mathcal{I}$ - name for a Borel set B from ideal $\mathcal{I}$. Let $\dot{r}$ be a canonical name for the generic real. Since some p in G forces that $\dot{B}$ is in $\mathcal{I}$ we can find a Borel set $C \subseteq \mathcal{X} \times \mathcal{X}$ in the ideal $\mathcal{I}$ and coded in the ground model, such that in $V[G]$ $B = C_{\dot{r}_G}$ (see for example [13]).

Now by the Fubini property:

$$\{x : \ C^x \notin \mathcal{I}\} \in \mathcal{I}.$$

If $B \cap \mathcal{X}^V$ is empty, there is nothing to prove. Thus, assume $B \cap \mathcal{X}^V$ is nonempty. Then consider any $x \in B \cap \mathcal{X}^V$. Then $V[G] \models x \in B$

$$0 < \|x \in \dot{B}\| = \|x \in C_{\dot{r}}\| = \|(\dot{r}, x) \in C\| = \|\dot{r} \in C^x\| = [C^x]_\mathcal{I}$$

Then we have:

$$B \cap \mathscr{X}^V \subseteq \{x : C^x \notin \mathcal{I}\}.$$

Now by the Fubini property we see that the set $\{x : C^x \notin \mathcal{I}\}$ is in ideal $\mathcal{I}$. But the last set is coded in ground model because the set C was coded in V. □

Theorem 3.1. *Assume that $\omega < \kappa$ and $\mathcal{I}, \mathcal{J}$ are c.c.c. and have the Fubini property. Suppose that $\mathbb{P}_\mathcal{I} = \mathrm{Borel}(\mathscr{X}) \setminus \mathcal{I}$ and $\mathbb{P}_\mathcal{J} = \mathrm{Borel}(\mathscr{X}) \setminus \mathcal{J}$ are definable forcings. Then $\mathbb{P}_\mathcal{I}$ preserves the $(\kappa, \mathcal{I}, \mathcal{J})$ - Luzin set property.*

Proof. Let L be a $(\kappa, \mathcal{I}, \mathcal{J})$ - Luzin set in V. Let G be a $\mathbb{P}_\mathcal{I}$ – generic over V. In $V[G]$ take any $B \in \mathcal{I}$. Then $L \cap B \cap V = L \cap B$ but by Lemma 3.1 there exists a set $D \in \mathcal{I} \cap V$ such that $L \cap B \subseteq L \cap D \in \mathcal{I}$

so $L \cap D \in \mathcal{J}$ in V by definition of L. Finally, by the absolutness of Borel codes from ideal $\mathcal{J}$ we have

$$L \cap B \subseteq L \cap D \in \mathcal{J} \text{ in } V[G].$$

$\square$

Remark 3.1. *Let $\mathbb{P}$ be the random forcing and let us consider any Sierpiński set S in the ground model V. Then if we take an any null set A in an any generic extension $V^{\mathbb{P}}$ then one can find a null set $B \in V$ whith the property $A \cap V \subset B$. Then $S \cap B$ is countable and then $S \cap A$ is countable. Thus the set S is also Sierpiński set in the universe $V^{\mathbb{P}}$. Analogously, let observe that the Cohen forcing is preserving of being a Luzin set.*

Theorem 3.2. *Let $(\mathbb{P}, \leq)$ be a forcing notion such that*

$$\{B : B \in \mathcal{I} \cap \text{Borel}(\mathscr{X}), B \textit{ is coded in } V\}$$

is a base for $\mathcal{I}$ in $V^{\mathbb{P}}[G]$. Assume that Borel codes for sets from ideals $\mathcal{I}, \mathcal{J}$ are absolute. Then $(\mathbb{P}, \leq)$ preserves being a $(\mathcal{I}, \mathcal{J})$ - Luzin sets.

Proof. Let L be a $(\mathcal{I}, \mathcal{J})$ - Luzin set in ground model V. We will show that $V^{\mathbb{P}}[G] \models L$ is a $(\mathcal{I}, \mathcal{J})$ - Luzin set.

Let us work in $V^{\mathbb{P}}[G]$. Fix $I \in \mathcal{I}$. $\mathcal{I}$ has Borel base consisting of sets coded in V. So, there exists $b \in \omega^\omega \cap V$ such that $I \subseteq \#b \in \mathcal{I}$.

By absoluteness of Borel codes from $\mathcal{I}$ we have that $V \models \#b \in \mathcal{I}$. L is a $(\mathcal{I}, \mathcal{J})$ - Luzin set in the model V. So, there is $c \in \omega^\omega \cap V$ which codes Borel set from the ideal $\mathcal{J}$ such that $V \models L \cap \#b \subseteq \#c$. By absoluteness of Borel codes from $\mathcal{J}$ we get that

$$V^{\mathbb{P}}[G] \models L \cap B \subseteq L \cap \#b \subseteq \#c \in \mathcal{J},$$

what proves that L is a $(\mathcal{I}, \mathcal{J})$ - Luzin set in generic extension. $\square$

The above theorem gives us a series of corollaries.

Corollary 3.1. *Let $(\mathbb{P}, \leq)$ be any forcing notion which does not change the reals i. e. $(\omega^\omega)^V = (\omega^\omega)^{V^{\mathbb{P}}[G]}$. Assume that Borel codes for sets from ideals $\mathcal{I}, \mathcal{J}$ are absolute. Then $(\mathbb{P}, \leq)$ preserves being $(\mathcal{I}, \mathcal{J})$ - Luzin sets.*

Corollary 3.2. *Assume that $(\mathbb{P}, \leq)$ is a σ-closed forcing and Borel codes for sets from ideals $\mathcal{I}, \mathcal{J}$ are absolute. Then $(\mathbb{P}, \leq)$ preserves $(\mathcal{I}, \mathcal{J})$ - Luzin sets.*

Corollary 3.3. *Let $\lambda \in On$ be an ordinal number. Let $\mathbb{P}_\lambda = \langle (P_\alpha, \dot{Q}_\alpha) : \alpha < \lambda \rangle$ be iterated forcing with countable support. Suppose that*

(1) *for any $\alpha < \lambda$ $P_\alpha \Vdash \dot{Q}_\alpha - \sigma$ closed ,*
(2) *Borel codes for sets from ideals $\mathcal{I}, \mathcal{J}$ are absolute,*

then $\mathbb{P}_\lambda$ preserve $(\mathcal{I}, \mathcal{J})$ - Luzin sets.

Proof. Our forcing $\mathbb{P}_\lambda$ is σ-closed because it is countable support iteration of σ -closed forcings. So, we can apply Corollary 3.2 to finish the proof. □

Now, let us consider some properties of countable support iteration connected with preservation of some relation. We will follow notation given by Goldstern (see [6]).

First, let us consider measure case. Let Ω be a family of clopen sets of Cantor space 2^ω and

$$C^{random} = \{f \in \Omega^\omega : (\forall n \in \omega)\mu(f(n)) < 2^{-n}\}$$

with discrete topology. If $f \in C^{random}$ then let us define the following set $A_f = \bigcap_{n\in\omega} \bigcup_{k\geq n} f(k)$.

Now, we are ready to define the following relation $\sqsubseteq^{random} = \bigcup_{n\in\omega} \sqsubseteq_n^{random}$ where

$$(\forall f \in C^{random})(\forall g \in 2^\omega)(f \sqsubseteq_n^{random} g \longleftrightarrow (\forall k \geq n)\ g \notin f(k)).$$

Definition of the notion of preservation of relation $\sqsubseteq^{random}$ by forcing notion $(\mathbb{P}, \leq)$ can be found in paper [6]. Let us focus on the following consequence of that definition.

Fact 3.1 (Goldstern)**.** *If* $(\mathbb{P}, \leq)$ *preserves* $\sqsubseteq^{random}$ *then* $\mathbb{P} \Vdash \mu^*(2^\omega \cap V) = 1$.

Now, we say that forcing notion $\mathbb{P}$ preserves outer measure iff $\mathbb{P}$ preserves $\sqsubseteq^{random}$.

It is well known that Laver forcing preserves some stronger property than $\sqsubseteq^{random}$ (see [7]). So, Laver forcing preserves outer measure.

In [6] we can find the following theorem:

Theorem 3.3 (Goldstern)**.** *Let* $\mathbb{P}_\lambda = ((P_\alpha, Q_\alpha) : \alpha < \gamma)$ *be any countable support iteration such that*

$$(\forall \alpha < \gamma)\ P_\alpha \Vdash Q_\alpha \text{ preserves } \sqsubseteq^{random}$$

then $\mathbb{P}_\gamma$ *preserves the relation* $\sqsubseteq^{random}$.

Theorem 3.4. *Assume that* $\mathbb{P}$ *is a forcing notion which preserves* $\sqsubseteq^{random}$. *Then* $\mathbb{P}$ *preserves being an* $(\mathbb{L}, \mathbb{K})$*-Luzin set.*

Proof. Assume that $V \models L$ is a $(\mathbb{L}, \mathbb{K})$-Luzin set. Let us work in $V^{\mathbb{P}}[G]$. Take any null set $A \in \mathbb{L}$. Then there is a null set B in ground model such that $A \cap V \subseteq B$.

Indeed, let us assume that there is no such $B \in V$. Then without loss of generality $(2^\omega \setminus A) \cap V \in \mathbb{L}$. But $A \in \mathbb{L}$ then we have that $2^\omega \cap V \subset A \cup ((2^\omega \setminus A) \cap V)$ which is a null set. But by Fact 3.1 $\mu^*(2^\omega \cap V) = 1$. So we have a contradiction.

Then the intersection $A \cap L \subseteq B \cap L \in \mathbb{K}$ is a meager set in ground model. Then by absolutnes of Borel codes of meager sets the set $A \cap L$ is a meager set, finishing the proof. □

Remark 3.2. *In the constructible universe L let us consider the countable support forcing iteration $P_{\omega_2} = ((P_\alpha, Q_\alpha) : \alpha < \omega_2)$ of the length ω_2 as follows, for any $\alpha < \omega_2$*

- *if α is even then $P_\alpha \Vdash$ "Q_α is random forcing",*
- *if α is odd then $P_\alpha \Vdash$ "Q_α is Laver forcing".*

Previously we noticed that both random and Laver forcing, preserves $\sqsubseteq^{random}$ and then by Theorem 3.3 P_{ω_2} preserves relation $\sqsubseteq^{random}$. By Theorem 3.4 the $(\mathbb{L}, \mathbb{K})$-Luzin sets are preserved by our iteration P_{ω_2}. Moreover, in the generic extension we have $cov(\mathbb{L}) = \omega_2$ and $2^\omega = \omega_2$ (for details see [6]*).*

Assume that in the ground model A is a $(\mathbb{L}, \mathbb{K})$-Luzin set with outer measure equal to one. Then in generic extension it has outer measure one and $|A| = \omega_1$. So, it does not contain any Lebesgue positive Borel set. Thus A is completely $\mathbb{L}$-nonmeasurable set.

In particular, if A is a Sierpiński set in the constructible universe L with outer measure equal to one. Then after above iteration the set A is completely $\mathbb{L}$-nonmeasurable set.

The analogous machinery can be used for ideal of meager sets $\mathbb{K}$. Let us recall the necessary definitions (see [6]).

Let C^{Cohen} be a set of all functions from $\omega^{<\omega}$ into itself. Then $\sqsubseteq^{Cohen} = \bigcup_{n\in\omega} \sqsubseteq^{Cohen}_n$ and for any $n \in \omega$ let

$$(\forall f \in C^{Cohen})(\forall g \in \omega^\omega)(f \sqsubseteq^{Cohen}_n g \text{ iff } (\forall k < n)(g \restriction k^\frown f(g \restriction k) \subseteq g)).$$

Then finally we have the following theorem:

Theorem 3.5. *Assume that $\mathbb{P}$ is a forcing notion which preserves $\sqsubseteq^{Cohen}$. Then $\mathbb{P}$ preserves being a $(\mathbb{K}, \mathbb{L})$-Luzin set.*

The another preservation theorem which is due to Shelah (see [11] and also [12]) is as follows

Theorem 3.6 (Shelah)**.** *Let $\mathbb{P}_\lambda = ((P_\alpha, \dot{Q}_\alpha) : \alpha < \lambda)$ be any countable support iteration such that $(\forall \alpha < \gamma)$ $P_\alpha \Vdash Q_\alpha$ is proper and*

$$P_\alpha \Vdash Q_\alpha \Vdash \textit{every new open dense set contains old open dense set}$$

then $\mathbb{P}_\lambda \Vdash$ every new open dense contains old open dense set.

We can easily derive

Corollary 3.4. *Let $\mathbb{P}_\lambda = ((P_\alpha, \dot{Q}_\alpha) : \alpha < \lambda)$ be any countable support iteration such that $(\forall \alpha < \lambda)$ $P_\alpha \Vdash Q_\alpha$ is proper and*

$$P_\alpha \Vdash Q_\alpha \Vdash \textit{every new open dense set contains old open dense set}$$

Then $\mathbb{P}_\lambda$ preserves being a $(\mathbb{K}, \mathbb{L})$-Luzin set.

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