On the splitting number and Mazurkiewicz's theorem

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Abstract. We give a new proof of Mazurkiewicz's theorem about bounded sequences of Borel functions. In this proof we use Shoenfield's absoluteness theorem for $\sum_2^1$ - sentences and one characterization of some class of sequentially compact topological spaces which involves the splitting number.

1. Introduction

We use standard set theoretical notation. By ω we denote the set of natural numbers. By $[X]^\omega$ we denote the family of all infinite subsets of a set X. The cardinality of a set X we denote by $|X|$. By κ, λ we always denote infinite cardinal numbers.

By $[0,1]$ we denote the unit interval of the real line. For an infinite cardinal number κ by $\{0,1\}^\kappa$ we denote the generalized Cantor set of length κ. Similarly $[0,1]^\kappa$ denotes the generalized Tichonov cube of length κ. By $Perf([0,1])$ we denote the Polish space of all non-empty perfect subsets of the interval $[0,1]$ with the Hausdorff metric. We treat the set $[\omega]^\omega$ as a Polish space, since we may identify this set with a G_δ subset of the classical Cantor set $\{0,1\}^\omega$.

Let us recall that the *splitting* number $\mathbf{s}$ (see [**4**]) is the least cardinal number such that there exists a family $\mathcal{F}$ of infinite subsets of ω such that $(\forall A \in [\omega]^\omega)(\exists S \in \mathcal{F})(|A \cap S| = |A \setminus S| = \omega)$. It is well known that $\omega < \mathbf{s} \leq 2^\omega$ and that $\mathbf{s}$ is a relatively small cardinal number. Moreover, Martin's Axiom implies that $\mathbf{s} = 2^\omega$. This implies that each transitive model of the theory ZFC can be extended, via a forcing extension, to a model of theory $ZFC \cup \{\mathbf{s} > \aleph_1\}$.

A topological space is *sequentially compact* if each sequence of its elements has a convergent subsequence. It is known that a countable product of sequentially compact spaces is sequentially compact and that a continuous image of sequentially compact space is sequentially compact, too (see [**1**]). We shall use the following characterization of the cardinal number $\mathbf{s}$ (see [**4**]):

$$\mathbf{s} = \min\{\kappa : \{0,1\}^\kappa \text{ is not sequentially compact}\}.$$

By a canonical Polish space we understood a countable product of spaces $\{0,1\}^\omega$, $[0,1]$, $Perf([0,1])$ and so on. A sentence φ is a $\sum_2^1$ - sentence if for some canonical Polish spaces X, Y and some Borel subset $B \subseteq X \times Y$ we have

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$\varphi = (\exists x \in X)(\forall y \in Y)((x,y) \in B)$. Spaces X, Y and the set B are called "parameters" of the sentence φ. We shall use the following classical theorem about absoluteness of $\sum_2^1$ - sentences (see [**3**]):

THEOREM 1. *(Shoenfield) Suppose that $M \subseteq N$ are transitive models of the theory ZF such that $\omega_1^N \subset M$. Let φ be $\sum_2^1$ - sentences with parameters from the model M. Then φ holds in the model M if and only if φ holds in the model N.*

Notice that if the model N is a generic extension of the transitive model M then both models M and N have the same ordinal numbers, so the inclusion $\omega_1^N \subset M$ trivially holds.

2. Proof of Mazurkiewicz's theorem

We start our consideration with one probably well known characterization of the splitting number.

LEMMA 1. *The following three cardinal numbers are the same:*
1. $\mathbf{s} = \min\{\kappa : \{0,1\}^\kappa$ *is not sequentially compact*$\}$,
2. $\mathbf{s}' = \min\{\kappa : (\{0,1\}^\omega)^\kappa$ *is not sequentially compact*$\}$,
3. $\mathbf{s}'' = \min\{\kappa : [0,1]^\kappa$ *is not sequentially compact*$\}$.

PROOF. Suppose that $\lambda < \mathbf{s}$. Then the space $\{0,1\}^\lambda$ is sequentially compact. But the space $(\{0,1\}^\omega)^\lambda \backsimeq (\{0,1\}^\lambda)^\omega$ is a product of countably many sequentially compact spaces, so it is sequentially compact, too. This shows that $\mathbf{s} \leq \mathbf{s}'$. Suppose now that $\lambda < \mathbf{s}'$. Then the space $(\{0,1\}^\omega)^\lambda$ is sequentially compact. Therefore the space $[0,1]^\lambda$, as a continuous image of the space $(\{0,1\}^\omega)^\lambda$, is sequentially compact, too. This shows that $\mathbf{s}' \leq \mathbf{s}''$. Finally, notice that if $\{f_n\}_{n\in\omega}$ is a sequence of elements of the space $\{0,1\}^\kappa$ without any convergent subsequence then the same sequence $\{f_n\}_{n\in\omega}$, treated as a sequence of element of the space $[0,1]^\kappa$, has no any convergent subsequence. This shows that $\mathbf{s}'' \leq \mathbf{s}$. ∎

Now we formulate and give a new proof of one theorem about sequences of bounded Borel functions proved in 1932 by Mazurkiewicz in [**2**].

THEOREM 2. *(Mazurkiewicz) Let $\{f_n\}_{n\in\omega}$ be a sequence of Borel functions from $[0,1]$ to $[0,1]$. Then there exists a non-empty perfect subset P of $[0,1]$ and a subsequence $\{f_{n_k}\}_{k\in\omega}$ which is pointwise convergent on the set P.*

PROOF. Let $\{f_n\}_{n\in\omega}$ be a sequence of Borel functions from $[0,1]$ to $[0,1]$. Let V' be a generic extension of the universe V such that $V' \models (\mathbf{s} > \aleph_1)$.

For a while we shall work in the universe V'. We choose an arbitrary set $T \subset [0,1]$ of cardinality $\aleph_1$ and consider the sequence $\{f_n \restriction T\}_{n\in\omega}$. Since the inequality $|T| < \mathbf{s}$ holds, by Lemma 1, there exists an infinite set $A \subset \omega$ such that the sequence $\{f_n \restriction T\}_{n\in A}$ is pointwise convergent on the whole set T. Notice that the set

$$C = \{x \in [0,1] : \{f_n(x)\}_{n\in A} \text{is convergent}\}$$

is Borel and contains the set T. But T is uncountable, therefore the set C contains a non-empty perfect set. Therefore the sentence

$$\varphi = (\exists A \in [\omega]^\omega)(\exists P \in \mathit{Perf}([0,1]))(\forall x \in P)(\{f_n(x)\}_{n\in A} \text{is convergent})$$

holds in the universe V'. But φ is a $\sum_2^1$ - sentence with parameters from the universe V and so, by Shoenfield's absoluteness theorem, φ holds in the universe V, too. ∎

References

[1] Engelking, R., *Outline of general topology*, PWN, Warsaw;North-Holland, Amsterdam, **1974**

[2] Mazurkiewicz, S., *Sur les suites de functionc continues*, Fund. Math., 18, **1932**, pp. 114-117

[3] Shoenfield, J., *The problem of predictavity*, Essays on the Foundations of Mathematics, Jerusalem, **1961**, p. 132-139

[4] van Douwen, E. K., *The integers and topology,* Handbook of Set Theoretical Topology (K. Kunen and J. Vaughan, eds), North-Holland, Amsterdam **1984**, pp. 111-167

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