On completely nonmeasurable unions

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Abstract. Assume that there is no quasi-measurable cardinal not greater than 2^ω. We show that for a c.c.c. σ-ideal $\mathbb{I}$ with a Borel base of subsets of an uncountable Polish space, if $\mathcal{A}$ is a point-finite family of subsets from $\mathbb{I}$ then there is a subfamily of $\mathcal{A}$ whose union is completely nonmeasurable i.e. its intersection with every non-small Borel set does not belong to the σ-field generated by Borel sets and the ideal $\mathbb{I}$. This result is a generalization of Four Poles Theorem (see [**1**]) and result from [**3**].

1. Introduction

It is known that for a partition $\mathcal{A}$ of the real line consisting of sets of Lebesgue measure zero, the union of some of these sets is Lebesgue nonmeasurable. Analogous result is known for the sets of the first category (the Lebesgue measurability is then replaced by having the Baire property) (see [**7**]). Actually, this result remains true, if in the above statement, the real line is replaced by any Polish space, the σ-ideals of sets of Lebesgue measure zero or of sets of the first category are replaced by any σ-ideal I with a Borel base and instead of assuming that the family $\mathcal{A}$ of sets of an ideal is a partition of the space, we assume that $\mathcal{A}$ is point-finite and its union is not in I. The conclusion says now that there exists a subfamily $\mathcal{A}'$ of $\mathcal{A}$ such that the union of its sets is not in the σ-algebra generated by the σ-algebra of Borel sets and I (see [**1**]). This result is called in the literature Four Poles Theorem.

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It is known that within ZFC it is not possible to replace the assumption that the family $\mathcal{A}$ is point-finite even by the one saying that $\mathcal{A}$ is point-countable (see [**4**]).

In various cases it is possible to obtain more than nonmeasurability of the union of a subfamily of $\mathcal{A}$. Namely, the intersection of this union with any measurable set that is not in I is nonmeasurable (recall, the measurability is understood here in the sense of belonging to the σ-algebra generated by the family of Borel sets and I). Such strong conclusion can be obtained for the ideal of first Baire category sets under the assumption that $\mathcal{A}$ is a partition, but without assuming anything about the regularity of the elements of $\mathcal{A}$ (see [**3**]).

Some related topics are presented in [**6**], chapter 14. Namely, the problem of the existance of the family of first category sets whose union do not posses Baire property is discussed for general topological spaces of second Baire category.

In this paper we show how to obtain complete nonmeasurability of the union of a subfamily of $\mathcal{A}$ assuming that $\mathcal{A}$ is point-finite family. We also use some set-theoretic assumptions. Namely, we assume that there is no quasi-measurable cardinal not greater than 2^ω. (Recall that κ is quasi-measurable if there exists a κ-additive ideal I of subsets of κ such that the Boolean algebra $P(\kappa)/I$ satisfies countable chain condition.) By the Ulam theorem (see [**5**]) every quasi-measurable cardinal is weakly inaccessible, so it is a large cardinal. In particular, we obtain results for the ideal of Lebesgue measure zero sets and first Baire category sets.

2. Definitions and notations

The cardinality of a set A is denoted by $|A|$. Cardinal numbers will usually be denoted by κ and λ. The family of all subsets of cardinality not bigger than κ of a set A is denoted by $[A]^{\leq\kappa}$. The set of real numbers is denoted by $\mathbb{R}$. An *ideal* I of subsets of a set X is a family of subsets of X which is closed under finite unions and taking subsets and such that $[X]^{<\omega} \subseteq I$. A family of sets is a *σ-ideal* if it is an ideal and is closed under countable unions.

For a topological space T, by $\mathcal{B}_T$ we denote the family of Borel subsets of T. If I is an ideal of subsets of a set X and $\mathcal{S}$ is a field of subsets of X, then by $\mathcal{S}[I]$ we denote the field generated by $\mathcal{S} \cup I$. If I is a σ-ideal and $\mathcal{S}$ is a σ-field then $\mathcal{S}[I]$ is a σ-field, too.

Let T be an uncountable Polish topological space with σ-finite Borel measure. The σ-ideal of all meagre sets of T is denoted by $\mathbb{K}_T$ and the σ-ideal of measure zero sets is denoted by $\mathbb{L}_T$. The Lebesgue measure on the real line is denoted by λ. Symbol λ_* denotes the corresponding inner measure. The σ-ideal of Lebesgue measure zero subsets of $\mathbb{R}$ will be denoted by $\mathbb{L}$ and the σ-ideal of sets of the first Baire category in $\mathbb{R}$ will be denoted by $\mathbb{K}$. Then $\mathcal{B}_{\mathbb{R}}[\mathbb{L}]$ is the σ-field of Lebesgue measurable subsets of $\mathbb{R}$ and $\mathcal{B}_{\mathbb{R}}[\mathbb{K}]$ is the σ-field of subsets of $\mathbb{R}$ with the Baire property.

If I is an ideal of subsets of a topological space T then we say that the ideal I has a Borel base if for each set $X \in I$ there exists a set $Y \in \mathcal{B}_T$ such that $X \subseteq Y$ and $Y \in I$. The two classical ideals $\mathbb{K}$ and $\mathbb{L}$ have Borel bases.

Let $\mathbb{B}$ be a complete Boolean algebra. We say that $\mathbb{B}$ satisfy c.c.c. (countable chain condition) if every antichain of elements of $\mathbb{B}$ is countable. Boolean algebras $\mathcal{B}_T[\mathbb{K}_T]/\mathbb{K}_T$ and $\mathcal{B}_T[\mathbb{L}_T]/\mathbb{L}_T$ satisfies c.c.c.

We say that the cardinal number κ is *quasi-measurable* if there exists κ-additive ideal $\mathcal{I}$ of subsets of κ such that the Boolean algebra $P(\kappa)/\mathcal{I}$ satisfies c.c.c. Cardinal κ is *weakly inaccessible* if κ is regular cardinal and for every cardinal $\lambda < \kappa$ we have that $\lambda^+ < \kappa$. Recall that every quasi-measurable cardinal is weakly inaccessible (see [**5**]).

3. Completely nonmeasurable unions

In this section T denotes an uncountable Polish space. Let us recall the following definition.

Definition 3.1. Let $\mathbb{I}$ be a σ-ideal of subsets of T with Borel base. Let $N \subseteq X \subseteq T$. We say that the set N is *completely* $\mathcal{B}_T[\mathbb{I}]$ *nonmeasurable* in X if

$$(\forall A \in \mathcal{B}_T)(A \cap X \notin \mathbb{I} \rightarrow (A \cap N \notin \mathbb{I}) \wedge (A \cap (X \setminus N) \notin \mathbb{I})).$$

In particular, $N \subseteq \mathbb{R}$ is completely $\mathcal{B}_{\mathbb{R}}[\mathbb{L}]$ nonmeasurable if $\lambda_*(N) = 0$ and $\lambda_*(\mathbb{R} \setminus N) = 0$. The definition of completely $\mathcal{B}_T[\mathbb{K}_T]$ nonmeasurability is equivalent to the definition of completely Baire nonmeasurability presented in [**3**] and [**8**].

Complete $\mathcal{B}_T[\mathbb{I}]$ nonmeasurability is also called being $\mathcal{B}_T[\mathbb{I}]$-Bernstein set (see [**3**]).

Definition 3.2. Let $\mathbb{I} \subseteq P(T)$ be a σ-ideal with Borel base such that the Boolean algebra $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c. Let A be any subset of T. By $[A]_{\mathbb{I}}$ we denote the Borel envelope of A, i.e. the minimal (in sense of the algebra $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$) Borel set containing A.

The set $[A]_{\mathbb{I}}$ is well defined since the algebra $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c. To find $[A]_{\mathbb{I}}$, take the maximal antichain $\mathcal{A}$ of $\mathbb{I}$-positive Borel sets which are disjoint from A. The family $\mathcal{A}$ is countable. So, its union $\bigcup\mathcal{A}$ is a Borel set. The complement $(\bigcup\mathcal{A})^c$ is the required envelope.

We use the following theorem (see [**8**]). We shall present the proof of it for readers convenience.

THEOREM 3.3. *Let $\mathbb{I}\subseteq P(T)$ be a σ-ideal with Borel base such that $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c. Let $\{A_\xi : \xi\in\omega_1\}$ be any family of subsets of T. Then we can find a family $\{I_\alpha\}_{\alpha\in\omega_1}$ of pairwise disjoint countable subsets of ω_1 such that for $\alpha<\beta<\omega_1$ we have that $[\bigcup_{\xi\in I_\alpha} A_\xi]_{\mathbb{I}} = [\bigcup_{\xi\in I_\beta} A_\xi]_{\mathbb{I}}$.*

PROOF. Fix a family $\{A_\xi : \xi\in\omega_1\}\subseteq P(T)$. For $\alpha<\beta<\omega_1$ let

$$A_\alpha^\beta = \bigcup\{A_\xi : \alpha<\xi<\beta\}.$$

Proceeding by transfinite induction, we construct a sequence $\{\alpha_\xi : \xi<\omega_1\}$ of ordinals less than ω_1 such that

$$(\forall\xi,\zeta,\eta\in\omega_1)((\xi<\zeta<\eta)\to[A_{\alpha_\xi}^{\alpha_\zeta}]_{\mathbb{I}} = [A_{\alpha_\xi}^{\alpha_\eta}]_{\mathbb{I}}).$$

In β-step consider a sequence $\{[A_{\alpha_\beta}^{\alpha}]_{\mathbb{I}} : 0<\alpha<\omega_1\}$. It is an increasing sequence. Since $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c. this sequence is constant from some point γ. Put $\alpha_{\beta+1}=\gamma$.

Now, consider a sequence $\{[A_{\alpha_\xi}^{\alpha_{\xi+1}}]_{\mathbb{I}} : \xi<\omega_1\}$. It is a decreasing sequence. Since $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c. this sequence is constant from some $\gamma<\omega_1$. For $\beta<\omega_1$ put $I_\beta=(\alpha_{\gamma+\beta},\alpha_{\gamma+\beta+1})$. The family $\{I_\beta\}_{\beta\in\omega_1}$ satisfies required condition. ☐

The next lemma is the only place where we use the non-existance of quasi-measurable cardinal not grater than the continuum.

LEMMA 3.4. *Assume that there is no quasi-measurable cardinal not greater than 2^ω. Let $\mathbb{I}$ be a σ-ideal of subsets of T with Borel base and such that $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c. Let $\mathcal{A}\subseteq\mathbb{I}$ be a point-finite family such that $\bigcup\mathcal{A}\notin\mathbb{I}$. Then there exists a family $\{\mathcal{A}_\alpha\}_{\alpha\in\omega_1}$ satisfying the following conditions*

(1) $(\forall\alpha<\omega_1)(\mathcal{A}_\alpha\subseteq\mathcal{A}\wedge\bigcup\mathcal{A}_\alpha\notin\mathbb{I})$,
(2) $(\forall\alpha<\beta<\omega_1)(\mathcal{A}_\alpha\cap\mathcal{A}_\beta=\emptyset)$,
(3) $(\forall\alpha,\beta<\omega_1)([\bigcup\mathcal{A}_\alpha]_{\mathbb{I}}=[\bigcup\mathcal{A}_\beta]_{\mathbb{I}})$.

PROOF. The family $\mathcal{A}$ is a point-finite family of subsets of uncountable Polish space. So, $|\mathcal{A}|\le 2^\omega$. Let $|\mathcal{A}|=\kappa$ and $\mathcal{A}=\{A_\xi\}_{\xi<\kappa}$. Consider the family $\mathcal{I}\subseteq P(\kappa)$ defined by the formula $X\in\mathcal{I}\leftrightarrow\bigcup_{\xi\in X}A_\xi\in\mathbb{I}$. $\mathcal{I}$ is a σ-ideal of subsets of κ. By the assumption of non-existance of quasi-measurable cardinal not greater than 2^ω, the algebra $P(\kappa)/\mathcal{I}$ is not c.c.c. So, we can find a family $\{\mathcal{A}'_\alpha\}_{\alpha<\omega_1}$ such that

(1) $(\forall\alpha<\omega_1)(\mathcal{A}'_\alpha\subseteq\mathcal{A}\wedge\bigcup\mathcal{A}'_\alpha\notin\mathbb{I})$,
(2) $(\forall\alpha<\beta<\omega_1)(\mathcal{A}'_\alpha\cap\mathcal{A}'_\beta=\emptyset)$.

Put $A_\alpha=\bigcup\mathcal{A}'_\alpha$. Using Theorem 3.3 we can find a family $\{I_\alpha\}_{\alpha\in\omega_1}$ such that for $\alpha<\beta<\omega_1$ we have that $[\bigcup_{\xi\in I_\alpha}A_\xi]_{\mathbb{I}}=[\bigcup_{\xi\in I_\beta}A_\xi]_{\mathbb{I}}$.

Put $\mathcal{A}_\alpha=\bigcup\{\mathcal{A}'_\xi:\xi\in I_\alpha\}$. The family $\{\mathcal{A}_\alpha\}_{\alpha<\omega_1}$ satisfies the required conditions. □

Lemma 3.5. *Let $\mathbb{I}$ be a σ-ideal of subsets of T with Borel base and such that $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c. Let $\mathcal{A}\subseteq P(T)$ be any point-finite family. Then there exists a subfamily $\mathcal{A}'\subseteq\mathcal{A}$ such that $|\mathcal{A}\setminus\mathcal{A}'|\leq\omega$ and*

$$(\forall B\in\mathcal{B}_T[\mathbb{I}])(\forall A\in\mathcal{A}')(B\cap\bigcup\mathcal{A}\notin\mathbb{I}\to\neg(B\cap\bigcup\mathcal{A}\subseteq B\cap A)).$$

Proof. The family $\mathcal{A}$ is point-finite. Let

$$B_n=\{x\in\bigcup\mathcal{A}:|\{A\in\mathcal{A}:x\in A\}|=n\}.$$

We have that

$$\bigcup\mathcal{A}=\bigcup\{B_n:n\in\omega\setminus\{0\}\}.$$

For each $n\in\omega$ let

$$\mathcal{A}_n=\{A\in\mathcal{A}:(\exists C\in\mathcal{B}_T[\mathbb{I}])(C\cap B_n\notin\mathbb{I}\wedge C\cap B_n\subseteq A\cap B_n)\}.$$

We show that $\mathcal{A}_n$ is countable for every $n\in\omega$. We do it by induction.

Case $n=1$ is straightforward since the algebra $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c.

Assume now that $\mathcal{A}_n$ is countable. Then $\mathcal{A}_{n+1}$ is countable. Indeed, take any element A of $\mathcal{A}_{n+1}$. By inductive assumption the family

$$\mathcal{A}_{n+1}[A]=\{A'\in\mathcal{A}_{n+1}:(\exists C\in\mathcal{B}_T[\mathbb{I}])(C\cap A\notin\mathbb{I}\wedge C\cap A\subseteq A'\cap A)\}$$

is countable. Take any maximal antichain $\tilde{\mathcal{A}}_{n+1}$ (in sense of the algebra $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$) of elements of $\mathcal{A}_{n+1}$. Then $\tilde{\mathcal{A}}_{n+1}$ is countable ($\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c.). and

$$\mathcal{A}_{n+1}=\bigcup\{\mathcal{A}_{n+1}[A]:A\in\tilde{\mathcal{A}}_{n+1}\}.$$

So, $|\mathcal{A}_{n+1}|\leq\omega$. Now, take $\mathcal{A}'=\mathcal{A}\setminus\bigcup_{n\in\omega}\mathcal{A}_n$. Naturally, $|\mathcal{A}\setminus\mathcal{A}'|\leq\omega$ and

$$(\forall B\in\mathcal{B}_T[\mathbb{I}])(\forall A\in\mathcal{A}')(B\cap\bigcup\mathcal{A}\notin\mathbb{I}\to\neg(B\cap\bigcup\mathcal{A}\subseteq B\cap A)).$$

□

Theorem 3.6. *Assume that there is no quasi-measurable cardinal not greater than 2^ω. Let $\mathbb{I}$ be a σ-ideal of subsets of T with Borel base and such that $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c. Let $\mathcal{A}\subseteq\mathbb{I}$ be a point-finite family* *such that* $\bigcup\mathcal{A} \notin \mathbb{I}$. *Then there exists a subfamily* $\mathcal{A}' \subseteq \mathcal{A}$ *such that* $\bigcup\mathcal{A}'$ *is completely* $\mathcal{B}_T[\mathbb{I}]$ *nonmeasurable in* $\bigcup\mathcal{A}$.

Proof. By transfinite induction we construct a family $\{B_\alpha\}$ of pairwise disjoint Borel sets and a family $\{\{\mathcal{A}^\alpha_\xi\}_{\xi\in\omega_1}\}$ of subfamilies of $\mathcal{A}$ satisfying the following conditions

(1) $B_\alpha \cap \bigcup\mathcal{A} \notin \mathbb{I}$,
(2) $(\forall\xi < \zeta < \omega_1)(\mathcal{A}^\alpha_\xi \cap \mathcal{A}^\alpha_\zeta = \emptyset)$,
(3) $(\forall\xi < \omega_1)([\bigcup\mathcal{A}^\alpha_\xi \setminus \bigcup_{\beta<\alpha} B_\beta]_{\mathbb{I}} = B_\alpha)$.

At α-step we consider the family $\mathcal{A}^\alpha = \{A \setminus \bigcup_{\xi<\alpha} B_\xi : A \in \mathcal{A}\}$. If $\bigcup\mathcal{A}^\alpha \in \mathbb{I}$ then we finish our construction. If $\bigcup\mathcal{A}^\alpha \notin \mathbb{I}$ then we use Lemma 3.4 to obtain a required family $\{\mathcal{A}^\alpha_\xi\}_{\xi\in\omega_1}$. We put $B_\alpha = [\bigcup\mathcal{A}^\alpha_0 \setminus \bigcup_{\zeta<\alpha} B_\zeta]_{\mathbb{I}}$.

Since $\mathcal{B}_T[\mathbb{I}]/\mathbb{I}$ satisfies c.c.c. the construction have to end up at some step $\gamma < \omega_1$.

Using transfinite induction we construct an increasing family $\{\mathcal{A}'_\alpha\}_{\alpha<\gamma}$ such that

(1) $\mathcal{A}'_\alpha \subseteq \mathcal{A}$,
(2) $\bigcup\mathcal{A}'_\alpha$ is completely $\mathcal{B}_T[\mathbb{I}]$ nonmeasurable in $\bigcup\mathcal{A} \cap \bigcup_{\zeta\leq\alpha} B_\zeta$.

We put $\mathcal{A}'_0 = \mathcal{A}^0_0$. Suppose that $\{\mathcal{A}_\zeta\}_{\zeta<\alpha}$ are defined. We consider a family $\{(\bigcup_{\zeta<\alpha} B_\zeta \cap \bigcup\mathcal{A}^\alpha_\xi) \setminus \bigcup_{\zeta<\alpha}\mathcal{A}'_\zeta : \xi < \omega_1\}$. It is a point-finite family. So, by Lemma 3.5 there exists $\xi < \omega_1$ (even ω_1 many of them) such that $(\bigcup_{\zeta<\alpha} B_\zeta \cap \bigcup\mathcal{A}^\alpha_\xi) \setminus \bigcup_{\zeta<\alpha}\mathcal{A}'_\zeta$ does not contain any $\mathbb{I}$-positive set of the form $C \cap (\bigcup_{\zeta<\alpha} B_\zeta \setminus \bigcup_{\zeta<\alpha}\mathcal{A}'_\zeta)$ where $C \in \mathcal{B}_T[\mathbb{I}]$. It means that $\mathcal{A}'_\alpha = \bigcup_{\zeta<\alpha}\mathcal{A}'_\xi \cup \mathcal{A}^\alpha_\xi$ satisfies required conditions.

We put $\mathcal{A}' = \bigcup_{\alpha<\gamma}\mathcal{A}'_\alpha$. It is clear that the set $\bigcup\mathcal{A}'$ is completely $\mathcal{B}_T[\mathbb{I}]$ nonmeasurable in $\bigcup\mathcal{A}$. □

Corollary 3.7. *Assume that there is no quasi-measurable cardinal not greater than* 2^ω. *Let* $\mathcal{A} \subseteq \mathbb{L}$ *be a point-finite family such that* $\bigcup\mathcal{A} = \mathbb{R}$. *Then there exists a subfamily* $\mathcal{A}' \subseteq \mathcal{A}$ *such that* $\lambda_*(\bigcup\mathcal{A}') = 0$ *and* $\lambda_*(\mathbb{R} \setminus \bigcup\mathcal{A}') = 0$.

Corollary 3.8. *Assume that there is no quasi-measurable cardinal not greater than* 2^ω. *Let* $\mathcal{A} \subseteq \mathbb{K}$ *be a point-finite family such that* $\bigcup\mathcal{A} = \mathbb{R}$. *Then there exists a subfamily* $\mathcal{A}' \subseteq \mathcal{A}$ *such that* $\bigcup\mathcal{A}'$ *is completely Baire nonmeasurable.*

Recall that the result from [**1**] was obtained within the standard axiomatic system of set theory and, moreover, does not require the c.c.c. of the quotient Boolean algebra. So, it is natural to pose the following question.

PROBLEM 1. *Is it possible to remove the assumption of non-existence of quasi-measurable cardinal not greater than 2^ω from Corollary 3.7 or Corollary 3.8 ?*

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