COMPLETELY NONMEASURABLE UNIONS

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ABSTRACT. Assume that no cardinal $\kappa < 2^\omega$ is quasi-measurable (κ is quasi-measurable if there exists a κ-additive ideal $\mathscr{I}$ of subsets of κ such that the Boolean algebra $P(\kappa)/\mathscr{I}$ satisfies c.c.c.). We show that for a metrizable separable space X and a proper c.c.c. σ-ideal $\mathbb{I}$ of subsets of X that has a Borel base, each point-finite cover $\mathscr{A} \subseteq \mathbb{I}$ of X contains uncountably many pairwise disjoint subfamilies $\mathscr{A}_\xi \subseteq \mathscr{A}$, $\xi < \omega_1$, with $\mathbb{I}$-Bernstein unions $\bigcup \mathscr{A}_\xi$ (a subset $A \subseteq X$ is $\mathbb{I}$-Bernstein if A and $X \setminus A$ meet each Borel $\mathbb{I}$-positive subset $B \subseteq X$). This result is a generalization of Four Poles Theorem (see [1]) and results from [2] and [4].

1. Notation and motivation

In this paper X will denote a metrizable separable space. Borel will denote the family of all Borel subsets of X. A family $\mathbb{I} \subseteq P(X)$ will be a σ-ideal of subsets of X with Borel base containing singletons. We will assume that $\mathbb{I}$ is a proper σ-ideal i.e $X \notin \mathbb{I}$. Let us recall that $\mathbb{I}$ has Borel base means that $(\forall I \in \mathbb{I})(\exists J \in \mathbb{I} \cap \text{Borel})(I \subseteq J)$. We say that a set $A \subseteq X$ is $\mathbb{I}$-*positive* if $A \notin \mathbb{I}$. We have the following cardinal coefficients:

$$\begin{aligned}
\operatorname{add}(\mathbb{I}) &= \min\{|\mathscr{C}| : \ \mathscr{C} \subseteq \mathbb{I},\ \bigcup \mathscr{C} \notin \mathbb{I}\},\\
\operatorname{cov}(\mathbb{I}) &= \min\{|\mathscr{C}| : \ \mathscr{C} \subseteq \mathbb{I},\ \bigcup \mathscr{C} = X\},\\
\operatorname{cov}_h(\mathbb{I}) &= \min\{|\mathscr{C}| : \ \mathscr{C} \subseteq \mathbb{I},\ (\exists B \in \text{Borel} \setminus \mathbb{I})(\bigcup \mathscr{C} \supseteq B)\},\\
\operatorname{cof}(\mathbb{I}) &= \min\{|\mathscr{C}| : \ \mathscr{C} \subseteq \mathbb{I}, (\forall I \in \mathbb{I})(\exists C \in \mathscr{C})(I \subseteq C)\}.
\end{aligned}$$

Similarly for a cover $\mathscr{A} \subseteq P(X)$ we can define

$$\begin{aligned}
\operatorname{add}(\mathscr{A}) &= \min\{|\mathscr{C}| : \ \mathscr{C} \subseteq \mathscr{A},\ \bigcup \mathscr{C} \notin \mathbb{I}\},\\
\operatorname{cov}_h^{\mathbb{I}}(\mathscr{A}) &= \min\{|\mathscr{C}| : \ \mathscr{C} \subseteq \mathscr{A},\ (\exists B \in \text{Borel} \setminus \mathbb{I})(\bigcup \mathscr{C} \supseteq B)\}.
\end{aligned}$$

Recall that the σ-ideal $\mathbb{I}$ has the *Steinhaus property* if for any two $\mathbb{I}$-positive Borel sets $A, B \in \text{Borel} \setminus \mathbb{I}$ the complex sum $A + B = \{a + b :$

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$a \in A, b \in B\}$ contains a nonempty open set. Let us remark that if the ideal $\mathbb{I}$ has the Steinhaus property, then $\mathrm{cov}_h(\mathbb{I}) = \mathrm{cov}(\mathbb{I})$.

Let us formulate the definition of the star which will be oftenly used in this paper.

Definition 1.1. *Assume that $\mathscr{A} \subseteq P(X)$ and $x \in X$. We say that $\mathscr{A}(x)$ is the $\mathscr{A}$-star of a point x if*

$$\mathscr{A}(x) = \{A \in \mathscr{A} : \quad x \in A\}.$$

We say that a family $\mathscr{A}$ is *point-finite* if it is a family with finite stars.

We start our consideration with the following theorem [1], known in literature as Four Poles Theorem.

Theorem 1.1 (Brzuchowski, Cichoń, Grzegorek, Ryll-Nardzewski)**.** *Let $\mathscr{A} \subseteq \mathbb{I}$ be a point-finite cover of X. Then there exists a subfamily $\mathscr{A}'$ such that $\bigcup \mathscr{A}'$ is not $\mathbb{I}$-measurable, i.e. does not belong to the σ-field generating by* Borel *and* $\mathbb{I}$.

There is a hypothesis stated by J. Cichoń saying that we can improve the conclusion of the above theorem to get $\bigcup \mathscr{A}'$ completely $\mathbb{I}$-nonmeasurable.

Definition 1.2. *A subset $A \subseteq X$ is called*

- *completely $\mathbb{I}$-nonmeasurable if for any $\mathbb{I}$-positive Borel subset $B \subseteq X$ both sets $A \cap B$ and $B \setminus A$ are $\mathbb{I}$-positive;*
- *$\mathbb{I}$-Bernstein if both A and $X \setminus A$ meet each $\mathbb{I}$-positive Borel subset of X.*

Let us observe that for the ideal $\mathbb{I}$ of countable subsets, $\mathbb{I}$-Bernstein sets are Bernstein in the classical sense.

Let us remark that definitions of completely $\mathbb{I}$-nonmeasurable set and $\mathbb{I}$-Bernstein are equivalent.

Proposition 1.1. *A subset $A \subseteq X$ is completely $\mathbb{I}$-nonmeasurable if and only if A is $\mathbb{I}$-Bernstein.*

We left the proof as an excercise to the reader.

Recall that $\mathbb{I}$ is c.c.c. if every family $\mathscr{A} \subseteq \mathrm{Borel} \setminus \mathbb{I}$ such that

$$(\forall A, A' \in \mathscr{A})(A = A' \vee A \cap A' \in \mathbb{I})$$

is at most countable.

Assume that $\mathscr{A} \subseteq \mathbb{I}$. Let $\mathscr{I}$ be an ideal on $P(\mathscr{A})$ associated with $\mathbb{I}$ in the following way

$$(\forall \mathscr{X} \in P(\mathscr{A}))(\mathscr{X} \in \mathscr{I} \longleftrightarrow \bigcup \mathscr{X} \in \mathbb{I}).$$

Then $W \subseteq P(\mathscr{A}) \setminus \mathscr{I}$ is an antichain in $P(\mathscr{A})/\mathscr{I}$ iff $(\forall a, b \in W)(a \neq b \longrightarrow a \cap b \in \mathscr{I})$. We say that $P(\mathscr{A})/\mathscr{I}$ is c.c.c. iff every antichain on $P(\mathscr{A})/\mathscr{I}$ is at most countable.

We say that the cardinal number κ is *quasi-measurable* if there exists κ-additive ideal $\mathscr{I}$ of subsets of κ such that the Boolean algebra $P(\kappa)/\mathscr{I}$ satisfies c.c.c. Cardinal κ is *weakly inaccessible* if κ is regular cardinal and for every cardinal $\lambda < \kappa$ we have that $\lambda^+ < \kappa$. Recall that every quasi-measurable cardinal is weakly inaccessible (see [3]), so it is a large cardinal.

Let us recall a result from [4].

Theorem 1.2 (Żeberski)**.** *Assume that no cardinal $\kappa \leq 2^\omega$ is quasi-measurable. Assume that $\mathbb{I}$ satisfies c.c.c. Let $\mathscr{A} \subseteq \mathbb{I}$ be a point-finite cover of X. Then there exists a subfamily $\mathscr{A}' \subseteq \mathscr{A}$ such that $\bigcup \mathscr{A}'$ is $\mathbb{I}$-Bernstein.*

The main result of this paper is the following theorem.

Theorem 1.3. *Assume that no cardinal $\kappa < 2^\omega$ is quasi-measurable. Assume that the ideal $\mathbb{I}$ is c.c.c. Let $\mathscr{A} \subseteq \mathbb{I}$ be a point-finite cover of X. Then there exist pairwise disjoint subfamilies $\mathscr{A}_\xi$, $\xi \in \omega_1$, of $\mathscr{A}$ such that each union $\bigcup \mathscr{A}_\xi$ is $\mathbb{I}$-Bernstein.*

2. Auxiliary Results

For a subset $D \subseteq X$ let $\lceil D \rceil_{\mathbb{I}}$ denote the set of minimal elements of the family $\{B \in \text{Borel} : D \subseteq_{\mathbb{I}} B\}$ partially preordered by the relation $A \subseteq_{\mathbb{I}} B$ (meaning that $A \setminus B \in \mathbb{I}$). If the ideal $\mathbb{I}$ is c.c.c., then $\lceil D \rceil_{\mathbb{I}}$ is not empty and thus contains a Borel set $B \supseteq_{\mathbb{I}} D$ such that $B \subseteq_{\mathbb{I}} B'$ for each Borel set $B' \supseteq_{\mathbb{I}} D$.

Let us recall three technical lemmas from [4] (Theorem 3.3, Lemma 3.4, Lemma 3.5).

Lemma 2.1 (Żeberski)**.** *If the ideal $\mathbb{I}$ is c.c.c., then for any uncountable family $\{A_\xi : \xi \in \omega_1\}$ of subsets of X there is an uncountable family $\{I_\alpha\}_{\alpha \in \omega_1}$ of pairwise disjoint countable subsets of ω_1 such that $\lceil \bigcup_{\xi \in I_\alpha} A_\xi \rceil_{\mathbb{I}} = \lceil \bigcup_{\xi \in I_\beta} A_\xi \rceil_{\mathbb{I}}$ for all $\alpha < \beta < \omega_1$.*

The next lemma is a reformulation of a result obtained in [4].

Lemma 2.2 (Żeberski)**.** *If the ideal $\mathbb{I}$ is c.c.c., and $\mathscr{A} \subseteq \mathbb{I}$ be a point-finite family such that $\bigcup \mathscr{A} \notin \mathbb{I}$ and the algebra $P(\mathscr{A})/\mathscr{I}$ is not c.c.c., then $\mathscr{A}$ contains uncountably many pairwise disjoint subfamilies $\mathscr{A}_\alpha$, $\alpha \in \omega_1$ such that $\lceil \bigcup \mathscr{A}_\alpha \rceil_{\mathbb{I}} = \lceil \bigcup \mathscr{A}_\beta \rceil_{\mathbb{I}} \neq \lceil \emptyset \rceil_{\mathbb{I}}$ for all $\alpha, \beta < \omega_1$.*

Lemma 2.3 (Żeberski)**.** *If the ideal $\mathbb{I}$ is c.c.c., then for each point-finite cover $\mathscr{A}$ of X the family $\mathscr{A}'$ of all sets $A \in \mathscr{A}$ containing an $\mathbb{I}$-positive Borel subset is at most countable.*

In paper [2] (Theorem 3.2) it is shown that if $\mathrm{cov}_h(\mathbb{I}) = \mathrm{cof}(\mathbb{I})$ and $\mathscr{A} \subseteq \mathbb{I}$ is a cover of X such that $\bigcup \mathscr{A}(x) \in \mathbb{I}$ for every $x \in X$, then there is a family $\mathscr{A}' \subseteq \mathscr{A}$ such that $\bigcup \mathscr{A}'$ is $\mathbb{I}$-Bernstein. This result can be generalized. Namely, we have the following theorem.

Theorem 2.1. *Let $\mathscr{A} \subseteq \mathbb{I}$ be a cover of X such that for any subset $D \subseteq X$ of cardinality $|D| < 2^\omega$ the union $\bigcup_{x\in D} \bigcup \mathscr{A}(x)$ contains no $\mathbb{I}$-positive Borel subset of X. Then $\mathscr{A}$ contains continuum many pairwise disjoint subfamilies $\mathscr{A}_\alpha$, $\alpha \in 2^\omega$, with $\mathbb{I}$-Bernstein unions $\bigcup \mathscr{A}_\alpha$.*

Proof. Let us enumerate the set of all Borel $\mathbb{I}$-positive sets $\mathrm{Borel} \setminus \mathbb{I} = \{B_\alpha :\ \alpha < 2^\omega\}$. By transfinite induction we will construct a sequence

$$((A_{\xi,\eta}, d_\xi) \in \mathscr{A} \times B_\xi :\ \ \xi, \eta < 2^\omega)$$

with the following conditions:

(1) $(\forall \xi, \eta < 2^\omega)(A_{\xi,\eta} \cap B_\xi \neq \emptyset)$,
(2) $\bigcup_{\xi,\eta<2^\omega} A_{\xi,\eta} \cap \{d_\xi : \xi < 2^\omega\} = \emptyset$,
(3) $(\forall \xi, \xi' < 2^\omega)(\forall \eta, \eta' < 2^\omega)(\eta \neq \eta' \longrightarrow A_{\xi,\eta} \neq A_{\xi',\eta'})$.

Let us fix $\alpha < 2^\omega$ and assume that we have defined the sequence

$$((A_{\xi,\eta}, d_\xi) \in \mathscr{A} \times B_\xi :\ \ \xi, \eta < \alpha)$$

with the following conditions:

(4) $(\forall \xi, \eta < \alpha)(A_{\xi,\eta} \cap B_\xi \neq \emptyset)$,
(5) $\bigcup_{\xi,\eta<\alpha} A_{\xi,\eta} \cap \{d_\xi : \xi < \alpha\} = \emptyset$,
(6) $(\forall \xi, \xi' < \alpha)(\forall \eta, \eta' < \alpha)(\eta \neq \eta' \longrightarrow A_{\xi,\eta} \neq A_{\xi',\eta'})$.

For every $\xi < \alpha$ let us consider the star $\mathscr{A}(d_\xi)$. By assumption the family $\bigcup_{\xi<\alpha} \mathscr{A}(d_\xi)$ does not cover any $\mathbb{I}$-positive Borel set. So, assumption guarantees that we can choose a set $\{A_{\alpha,\eta} \in \mathscr{A} :\ \ \eta < \alpha\} \subseteq \mathscr{A} \setminus \{A_{\xi,\eta} : \xi, \eta < \alpha\}$ of pairwise distinct sets such that

(7) $(\forall \eta < \alpha)(A_{\alpha,\eta} \cap B_\alpha \neq \emptyset)$,
(8) $(\forall \xi, \eta < \alpha)(d_\xi \notin A_{\alpha,\eta})$.

The same argument gives us the set $\{A_{\xi,\alpha} \in \mathscr{A} :\ \ \xi \leq \alpha\} \subseteq \mathscr{A} \setminus \{A_{\xi,\eta} : \xi \leq \alpha,\ \eta < \alpha\}$ of pairwise distinct sets with the following property:

$$(\forall \xi \leq \alpha)(A_{\xi,\alpha} \cap B_\xi \neq \emptyset\ \wedge\ A_{\xi,\alpha} \cap \{d_{\xi'} : \xi' < \alpha\} = \emptyset).$$

Once again by assumption we can find $d_\alpha \in B_\alpha$ such that $(\bigcup_{\xi,\eta\leq\alpha} A_{\xi,\eta}) \cap \{d_\alpha\} = \emptyset$. It finishes the α-step of our construction.

Now, let us put $\mathscr{A}_\eta = \{A_{\xi,\eta} \in \mathscr{A} :\ \ \xi < 2^\omega\}$ for any $\eta < 2^\omega$. The family $\{\mathscr{A}_\eta :\ \eta < 2^\omega\}$ fulfills the assertion of our Theorem. □

Corollary 2.1. *If* $\mathrm{cov}_h(\mathbb{I}) = 2^\omega$ *and* $\mathscr{A} \subseteq \mathbb{I}$ *is a cover of* X *such that* $|\mathscr{A}(x)| < cf(2^\omega)$ *for every* $x \in X$, *then there exists continuum many pairwise disjoint subfamilies* $\{\mathscr{A}_\alpha : \alpha \in 2^\omega\}$ *of the family* $\mathscr{A}$ *such that for every* $\alpha \in 2^\omega$ *the set* $\bigcup \mathscr{A}_\alpha$ *is completely* $\mathbb{I}$*-nonmeasurable.*

Theorem 2.2. *Assume that no cardinal* $\kappa < 2^\omega$ *is quasi-measurable. Let* $\mathscr{A} \subseteq \mathbb{I}$ *be a family with stars of size* $< 2^\omega$. *If* $\bigcup \mathscr{A} \notin \mathbb{I}$ *then* $P(\mathscr{A})/\mathscr{I}$ *is not c.c.c.*

Proof. Assume that $\mathscr{A} \subseteq \mathbb{I}$ satisfies the following conditions

(1) $\bigcup \mathscr{A} \notin \mathbb{I}$,
(2) $P(\mathscr{A})/\mathscr{I}$ is c.c.c.

Since 2^ω is the minimal possible quasi-measurable cardinal, $|\mathscr{A}| = 2^\omega$ and 2^ω is regular. Moreover $\mathrm{add}(\mathscr{A}) = 2^\omega$. By the regularity of the continuum and the fact that every star have size $< 2^\omega$ we get that $\mathrm{add}(\{\bigcup \mathscr{A}(x) : x \in X\}) = 2^\omega$. So the family $\mathscr{A}$ fulfils the assumptions of Theorem 2.1 (for $X = \bigcup \mathscr{A}$). By Theorem 2.1 there exists $\{\mathscr{C}_\alpha : \alpha < 2^\omega\}$ such that

(3) $\mathscr{C}_\alpha \subseteq \mathscr{A}$ for any $\alpha < 2^\omega$,
(4) $\forall \alpha < 2^\omega \ \bigcup \mathscr{C}_\alpha$ is completely $\mathbb{I}$-nonmeasurable,
(5) $\forall \alpha, \beta < 2^\omega \ \alpha \neq \beta \longrightarrow \mathscr{C}_\alpha \cap \mathscr{C}_\beta = \emptyset$.

In particular, the family $\{\mathscr{C}_\alpha : \alpha < 2^\omega\}$ forms an antichain in $P(\mathscr{A})/\mathscr{I}$, what gives a contradiction. □

Now, let us focus on the proof of main result.

Proof of Theorem 1.3. By transfinite induction we construct a family $\{B_\alpha\}$ of pairwise disjoint $\mathbb{I}$-positive Borel sets and a family $\{\{\mathscr{A}^\alpha_\xi\}_{\xi \in \omega_1}\}$ of subfamilies of $\mathscr{A}$ satisfying the following conditions

(1) $(\forall \xi < \zeta < \omega_1)(\mathscr{A}^\alpha_\xi \cap \mathscr{A}^\alpha_\zeta = \emptyset)$,
(2) $(\forall \xi < \omega_1)(B_\alpha \in \lceil \bigcup \mathscr{A}^\alpha_\xi \setminus \bigcup_{\beta<\alpha} B_\beta \rceil_{\mathbb{I}})$.

At α-step we consider the family $\mathscr{A}^\alpha = \{A \setminus \bigcup_{\xi<\alpha} B_\xi : A \in \mathscr{A} \setminus \bigcup_{\xi<\alpha} \mathscr{A}_\xi\}$. If $\bigcup \mathscr{A}^\alpha \in \mathbb{I}$ then we finish our construction. If $\bigcup \mathscr{A}^\alpha \notin \mathbb{I}$ then by Theorem 2.2 the algebra $P(\mathscr{A}^\alpha)/\mathscr{I}$ is not c.c.c. We use Lemma 2.2 to obtain a required family $\{\mathscr{A}^\alpha_\xi\}_{\xi \in \omega_1}$. We put B_α to be any member of $\lceil \bigcup \mathscr{A}^\alpha_0 \setminus \bigcup_{\zeta<\alpha} B_\zeta \rceil_{\mathbb{I}}$.

Since $\mathbb{I}$ satisfies c.c.c., the construction have to end up at some step $\gamma < \omega_1$.

Now put $\mathscr{A}'_\xi = \bigcup_{\alpha<\gamma} \mathscr{A}^\alpha_\xi$. By construction for each $\xi < \omega_1$ we have

$$\left\lceil \bigcup \mathscr{A}'_\xi \right\rceil_{\mathbb{I}} = \left\lceil \bigcup_{\alpha<\gamma} B_\alpha \right\rceil_{\mathbb{I}} = \lceil X \rceil_{\mathbb{I}}.$$

The family $\{\bigcup \mathscr{A}'_\xi : \xi \in \omega_1\}$ is point-finite because for every $x \in X$

$$\left|\left\{\bigcup \mathscr{A}'_\xi : x \in \bigcup \mathscr{A}'_\xi\right\}\right| \leq |\{A \in \mathscr{A} : x \in A\}| < \omega.$$

Now using Lemma 2.3 we can find a countable set $C \in [\omega_1]^\omega$ such that each member of the family $\{\bigcup \mathscr{A}'_\xi :\ \xi \in \omega_1 \setminus C\}$ does not contain any $\mathbb{I}$-positive Borel subset of X. So, the family $\{\mathscr{A}'_\xi :\ \xi \in \omega_1 \setminus C\}$ satisfies required conditions. □

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