SOME REMARKS ON TWO POINT SETS

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ABSTRACT. A subset of the plane is called two point set provided it intersects any line in exactly two points. Our results are the following. There exists a two point set which is a completely nonmeasurable Hamel base. There exists a two point set which is s_0 Marczewski and a two point set which is s-nonmeasurable. We are showing also some properties of a two point set and partial two point set.

1. INTRODUCTION

At the beginning of the XX century Mazurkiewicz in [Maz] constructed a set on the plane which meets any line in exactly two points. Any such set is called a two point set. Any two point set must be somehow complex, namely Larman in [L] show that it cannot be F_σ. It is a long standing open problem whether there is a Borel two point set (cf. [Mau]). The best known approximation to that problem is due to Miller who, under some set-theoretic assumption, proved that there is a coanalytic two point set [Mi]. Recently Schmerl proved in [S] that there is a two point set which can be covered by countably many circles, in particular there is a two point set which is meager and null.

In this note we show that there exist two point sets which have several properties. In theorem 1 we prove that it can be completely nonmeasurable. In theorem 3 we prove that it can be s_0 Marczewski. In theorem 4 we

1991 *Mathematics Subject Classification.* Primary: 03E75; Secondary: 03E35, 28A05, 15A03.

Key words and phrases. two point set, partial two point set, complete nonmeasurability, Hamel basis, s-Marczewski, s_0-Marczewski, s-nonmeasurability, Luzin set, Sierpiński set.

prove that it also can be s-nonmeasurable. It turns out that a two point set can be neither Luzin nor Sierpiński but partial two point set (i.e. a set that intersects any line in at most two points) can be Luzin or Sierpiński independently what will be shown in Theorem 6. In our constructions we modify that of a two point set given in [Ci, Theorem 6.1.2 p. 76-77].

Let us give a list of notions which will be used in a sequel. Let $A \subseteq \mathbb{R}^2$. We say that A is

- completely nonmeasurable if for every Borel set B of positive measure $A \cap B \neq \emptyset$ and $B \setminus A \neq \emptyset$;
- s_0 Marczewski iff for every perfect set P there exists a perfect set $Q \subseteq P$ such that $Q \cap A = \emptyset$;
- s Marczewski (s-measurable) iff for every perfect set P there exists perfect set $Q \subseteq P$ such that $Q \cap A = \emptyset$ or $Q \subset A$;
- *Bernstein* if for every perfect $P \subset \mathbb{R}^2$, $A \cap P \neq \emptyset$ and $P \setminus A \neq \emptyset$;
- *Luzin* if for every Borel set $B \in \mathbb{K}$, $|B \cap A| \leq \omega$;
- *strong Luzin* if for every Borel set $B \subseteq \mathbb{R}^2$, $|B \cap A| \leq \omega \leftrightarrow B \in \mathbb{K}$;
- *Sierpiński* if for every Borel set $B \in \mathbb{L}$, $|B \cap A| \leq \omega$;
- *strong Sierpiński* if for every Borel set $B \subseteq \mathbb{R}^2$, $|B \cap A| \leq \omega \leftrightarrow B \in \mathbb{L}$;

where $\mathbb{K}$ and $\mathbb{L}$ we denote the σ-algebras of meager and null subsets of the real plane respectively. Note that Bernstein sets can be viewed as completely nonmeasurable sets with respect to s-Marczewski σ-algebra.

2. A two point set that is completely nonmeasurable Hamel base

Theorem 1. *There exists a two point set $A \subset \mathbb{R}^2$, that is completely nonmeasurable Hamel base.*

Proof. Let $\{L_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all straight lines in the plane $\mathbb{R}^2$, let $\{B_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all Borel sets on a plane $\mathbb{R}^2$ of positive Lebesgue measure and let $\{h_\xi : \xi < \mathfrak{c}\}$ be a Hamel base of $\mathbb{R}^2$ over

$\mathbb{Q}$. We will define, by induction on $\xi < \mathfrak{c}$, the sequences $\{A_\xi : \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}^2$ and $\{x_\xi : \xi < \mathfrak{c}\}$ of elements of $\mathbb{R}^2$ such that for every $\xi < \mathfrak{c}$,

(1) $|A_\xi| < \omega$;
(2) $\bigcup_{\zeta \le \xi} A_\zeta$ does not have three collinear points;
(3) $\bigcup_{\zeta \le \xi} A_\zeta$ contains precisely two points of L_ξ;
(4) $B_\xi \cap \bigcup_{\zeta \le \xi} A_\zeta \neq \emptyset$;
(5) $x_\xi \in B_\xi$ and $A_\xi \cap \{x_\zeta : \zeta \le \xi\} = \emptyset$;
(6) $\bigcup_{\zeta \le \xi} A_\zeta$ is linearly independent over $\mathbb{Q}$;
(7) $h_\xi \in span_{\mathbb{Q}}(\bigcup_{\zeta \le \xi} A_\zeta)$.

Then, the set $A = \bigcup_{\xi < \mathfrak{c}} A_\xi$ will have desired property. Evidently, conditions (2) and (3) imply that the set A is a two point set, (5) implies the set A does not contain any set from $\{B_\xi : \xi < \mathfrak{c}\}$ and (4) makes sure it intersects all of them, so the set A is completely nonmeasurable; moreover conditions (6) and (7) imply that A is a Hamel base of $\mathbb{R}^2$ over $\mathbb{Q}$.

To end an inductive construction assume that for some $\xi < \mathfrak{c}$ sequences $\{A_\zeta : \zeta < \xi\}$ and $\{x_\zeta : \zeta < \xi\}$ are already defined. We will prove that there exist A_ξ and x_ξ satisfying (1)-(7). Let $\mathbb{B} = \bigcup_{\zeta < \xi} A_\zeta$ and $X = \bigcup_{\zeta < \xi} \{x_\zeta\}$. Clearly $|\mathbb{B}| < \mathfrak{c}$ and $|X| < \mathfrak{c}$. Let $\mathcal{G}$ be the family of all lines which meet $\mathbb{B}$ in exactly two points. Then $|\mathcal{G}| \le |\mathbb{B}^2| < \mathfrak{c}$. Consequently $|span_{\mathbb{Q}}(\mathbb{B})| < \mathfrak{c}$. We will define A_ξ and x_ξ in three steps. In each step we will focus on one of desired properties of A_ξ and x_ξ.

I. Let us focus on two point property. Note that (2) implies $L_\xi \cap \mathbb{B}$ has at most two points. Indeed, if ξ is a successor, i.e. $\xi = \xi_0 + 1$, then $\mathbb{B} = \bigcup_{\zeta \le \xi_0} A_\zeta$ and $|\mathbb{B} \cap L_\xi| > 2$ would led us to a contradiction with (2); if ξ is a limit ordinal, then $\mathbb{B} = \bigcup_{\zeta < \xi} (\bigcup_{\eta \le \zeta} A_\eta)$ and since the sequence $(\bigcup_{\eta \le \zeta} A_\eta)_{\zeta < \xi}$ is increasing, there exist $\zeta_0 < \xi$ such that $\mathbb{B} \cap L_\xi \subset \bigcup_{\eta \le \zeta_0} A_\eta$ and $|\mathbb{B} \cap L_\xi| > 2$ would led us to a contradiction with (2) as well. Consider following two cases:

(a) If $|L_\xi \cap \mathbb{B}| = 2$, then put $A_\xi^{(1)} = \emptyset$;

(b) If $|L_\xi \cap \mathbb{B}| < 2$ then $|L_\xi \cap l| \leq 1$ for any $l \in \mathcal{G}$. In fact, if there would exist $l \in \mathcal{G}$ with $|L_\xi \cap l| = 2$, then $L_\xi \in \mathcal{G}$ and by the definition of $\mathcal{G}$, we would obtain $|L_\xi \cap \mathbb{B}| = 2$, which gives a contradiction. Therefore $L_\xi \cap \bigcup \mathcal{G} = \bigcup_{l \in \mathcal{G}} l \cap L_\xi$ has cardinality less than $\mathfrak{c}$.

Choose

$$x^{(1)} \in L_\xi \setminus \left(span_{\mathbb{Q}}(\mathbb{B}) \cup X \cup \bigcup_{l \in \mathcal{G}} l \cap L_\xi \right)$$

and

$$y^{(1)} \in L_\xi \setminus \left(span_{\mathbb{Q}}(\mathbb{B} \cup \{x^{(1)}\}) \cup X \cup \bigcup_{l \in \mathcal{G}} l \cap L_\xi \right).$$

Put $A_\xi^{(1)} = \{x^{(1)}, y^{(1)}\}$ if $\mathbb{B} \cap L_\xi = \emptyset$ and put $A_\xi^{(1)} = \{x^{(1)}\}$ if $\mathbb{B} \cap L_\xi$ is a singleton. This can be done, because $|L_\xi| < \mathfrak{c}$,

$$\left| span_{\mathbb{Q}}(\mathbb{B}) \cup X \cup \bigcup_{l \in \mathcal{G}} l \cap L_\xi \right| < \mathfrak{c},$$

and

$$\left| span_{\mathbb{Q}}(\mathbb{B} \cup \{x^{(1)}\}) \cup X \cup \bigcup_{l \in \mathcal{G}} l \cap L_\xi \right| < \mathfrak{c}.$$

II. Let us focus on the complete nonmeasurability property

Let $\mathcal{G}'$ be the family of all lines which meet $\mathbb{B} \cup A_\xi^{(1)}$ in exactly two points. Then $|\mathcal{G}'| < \mathfrak{c}$ and $\mathcal{G} \subset \mathcal{G}'$. Since B_ξ is positive Borel set, $\mathfrak{c}$ many of its vertical sections are of positive measure in particular of cardinality $\mathfrak{c}$. Therefore we can find a vertical line L such that:

- $L \cap (\mathbb{B} \cup A_\xi^{(1)}) = \emptyset$;
- $|L \cap B_\xi| = \mathfrak{c}$.

It exists since $|\mathbb{B} \cup A_\xi^{(1)}| < \mathfrak{c}$. Notice that $|l \cap L| \leq 1$ for any $l \in \mathcal{G}'$.

Choose

$$x^{(2)} \in (L \cap B_\xi) \setminus \left(span_{\mathbb{Q}}(\mathbb{B} \cup A_\xi^{(1)}) \cup X \cup \bigcup_{l \in \mathcal{G}'} l \cap L \right),$$

and

$$y^{(2)} \in (L \cap B_\xi) \backslash \left(span_{\mathbb{Q}}(\mathbb{B} \cup A_\xi^{(1)} \cup \{x^{(2)}\}) \cup X \cup \bigcup_{l \in \mathcal{G}'} l \cap L \right),$$

and put $A_\xi^{(2)} = \{x^{(2)}, y^{(2)}\}$.

This can be done, since $|L \cap B_\xi| = \mathfrak{c}$,

$$\left| span_{\mathbb{Q}}(\mathbb{B} \cup A_\xi^{(1)}) \cup X \cup \bigcup_{l \in \mathcal{G}'} l \cap L \right| < \mathfrak{c},$$

and

$$\left| span_{\mathbb{Q}}(\mathbb{B} \cup A_\xi^{(1)} \cup \{x^{(2)}\}) \cup X \cup \bigcup_{l \in \mathcal{G}'} l \cap L \right| < \mathfrak{c}.$$

Choose $x_\xi \in (L \cap B_\xi) \backslash (X \cup \mathbb{B} \cup A_\xi^{(1)} \cup A_\xi^{(2)})$.

III. Let us focus on the condition (7). We have the following possibilities:

(a) If $h_\xi \in span_{\mathbb{Q}}(\mathbb{B} \cup A_\xi^{(1)} \cup A_\xi^{(2)})$, then (6) and (7) are satisfied by the set $\mathbb{B} \cup A_\xi^{(1)} \cup A_\xi^{(2)}$ and we are done by putting $A_\xi^{(3)} = \emptyset$;

(b) Assume that $h_\xi \notin span_{\mathbb{Q}}(\mathbb{B} \cup A_\xi^{(1)} \cup A_\xi^{(2)})$. Let $\mathcal{G}''$ be the family of all lines which meet $\mathbb{B} \cup A_\xi^{(1)} \cup A_\xi^{(2)}$ in exactly two points. Then $|\mathcal{G}''| < \mathfrak{c}$ and $\mathcal{G} \subset \mathcal{G}' \subset \mathcal{G}''$. Choose the line L parallel to h_ξ, with $L \cap (\mathbb{B} \cup A_\xi^{(1)} \cup A_\xi^{(2)}) = \emptyset$. This can be done, since there is $\mathfrak{c}$ many lines parallel to h_ξ and $|\mathbb{B} \cup A_\xi^{(1)} \cup A_\xi^{(2)}| < \mathfrak{c}$. Since $L \notin \mathcal{G}''$, then $|l \cap L| \leq 1$ for any $l \in \mathcal{G}''$.

Choose

$$x^{(3)} \in L \backslash \left(span_{\mathbb{Q}}(\mathbb{B} \cup A_\xi^{(1)} \cup A_\xi^{(2)} \cup \{h_\xi\}) \cup \left(\bigcup_{l \in \mathcal{G}''} l \cap L \right) \right),$$

such that

$$y^{(3)} = x^{(3)} + h_\xi \in L \backslash \left(span_{\mathbb{Q}}(\mathbb{B} \cup A_\xi^{(1)} \cup A_\xi^{(2)}) \cup (X \cup \{x_\xi\}) \cup \left(\bigcup_{l \in \mathcal{G}''} l \cap L \right) \right).$$

It is possible since

$$\left| L \backslash \left(span_{\mathbb{Q}}(\mathbb{B} \cup A_\xi^{(1)} \cup A_\xi^{(2)} \cup \{h_\xi\}) \cup \left(\bigcup_{l \in \mathcal{G}''} l \cap L \right) \right) \right| = \mathfrak{c}$$

and

$$\left|span_{\mathbb{Q}}(\mathbb{B}\cup A^{(1)}_\xi\cup A^{(2)}_\xi)\cup(X\cup\{x_\xi\})\cup\left(\bigcup_{l\in\mathcal{G}''} l\cap L\right)\right|<\mathfrak{c},$$

so there is less than $\mathfrak{c}$ many points x such that

$$y^{(3)}=x^{(3)}+h_\xi\in span_{\mathbb{Q}}(\mathbb{B}\cup A^{(1)}_\xi\cup A^{(2)}_\xi)\cup(X\cup\{x_\xi\})\cup\left(\bigcup_{l\in\mathcal{G}''} l\cap L\right).$$

Let $A^{(3)}_\xi=\{x^{(3)},y^{(3)}\}$.

Finally put $A_\xi=A^{(1)}_\xi\cup A^{(2)}_\xi\cup A^{(3)}_\xi$.

Conditions (1)-(7) are satisfied. □

3. A two point set that is s_0 Marczewski

For $p\in\mathbb{R}^2$ the symbol $||p||$ stands for the length of p. We know ([W]) that there exists a set $A\subseteq(0,1)$ which is s_0 and has cardinality $\mathfrak{c}$. We can get the following immediate proposition.

Proposition 2. *There is a set $B\subseteq(0,\infty)$ which is s_0 and satisfies following conditions*

(1) $|B\cap(0,\epsilon)|=\mathfrak{c}$, *for every $\epsilon>0$;*
(2) $|B\cap(T,\infty)|=\mathfrak{c}$, *for every $T>0$.*

Proof. Let $A\subseteq(0,1)$ be a s_0 set of cardinality $\mathfrak{c}$. Define B by the following formula:

$$B=\bigcup_{n<\omega} f_n[A]\cup g_n[B],$$

where $f_n:(0,1)\to(\frac{1}{n+2},\frac{1}{n+1})$ and $g_n:(0,1)\to(n+1,n+2)$ are increasing linear functions. The set B as a union of countably many s_0 sets is still s_0 and fulfils our requirements. □

Theorem 3. *There exists a two point set $A\subset\mathbb{R}^2$, that is s_0 Marczewski.*

Proof. Let $\{L_\xi:\xi<\mathfrak{c}\}$ be an enumeration of all straight lines in $\mathbb{R}^2$. Let $B\subseteq(0,\infty)$ be s_0 set from Proposition 2. We will define inductively a sequence $\{A_\xi:\xi<\mathfrak{c}\}$ of subsets of $\mathbb{R}^2$ such that for every $\xi<\mathfrak{c}$,

(1) $|A_\xi| < \omega$;

(2) $\bigcup_{\zeta\leq\xi} A_\zeta$ does not have three collinear points;

(3) $\bigcup_{\zeta\leq\xi} A_\zeta$ contains precisely two points of L_ξ;

(4) $||p|| \in B$ for every $p \in A_\xi$;

(5) for every different $p, q \in \bigcup_{\zeta\leq\xi} A_\zeta$ we have $||p| \neq ||q||$.

The existence of the sequence $\{A_\xi : \xi < \mathfrak{c}\}$ follows exactly in the same way as in Theorem 1. We repeat the proof of an existence of two point set in $\{p \in \mathbb{R}^2 : ||p|| \in B\}$ instead of $\mathbb{R}^2$. Here the key observation is that for each line L a set $\{p \in L : ||p|| \in B\}$ has cardinality continuum and therefore the construction can be done.

Put $A = \bigcup_{\xi<\mathfrak{c}} A_\xi$. Let us check that A is s_0. Take any perfect set $P \subseteq \mathbb{R}^2$. We have following cases

I. There is $r > 0$ such that $P_r = \{p \in P : ||p|| = r\}$ has cardinality continuum. Then $|P_r \cap A| \leq 1$ so we can find a perfect set $Q \subseteq P_r \setminus A \subseteq P \setminus A$ and we are done.

II. For every $r > 0$ a set P_r is countable. Then $\overline{P} = \{r > 0 : \ P_r \neq \emptyset\}$ is of size continuum. Moreover it is analytic, so $\overline{P}$ contains a perfect set $\overline{P}'$. Now, since B is s_0, we can find a perfect set $\overline{Q}$ such that $\overline{Q} \subseteq \overline{P}' \setminus B$. The set $Q = \{p \in P : ||p|| \in \overline{Q}\}$ is a required perfect set disjoint from A.

□

4. A two point set that is s-nonmeasurable

Let us note here that the unit circle is a partial two point set but it cannot be extended to a two point set. In [DM] and [DKM] it was investigated how small should be a subset of the unit circle to be extendable to a two point set. It turns out that sets of inner positive measure on the unit circle cannot be extended to two point sets. We show that there is a subset of the unit circle of full outer measure which can be extended to a two point set.

Theorem 4. *There exists a two point set $A \subset \mathbb{R}^2$, that is s-nonmeasurable. Moreover A contains a subset of the unit circle of full outer measure.*

Proof. Let us observe that if B is a Bernstein set in some uncountable closed set C then B is s-nonmeasurable. Moreover if a set D is such that $D \cup C = B$ then D also is s-nonmeasurable. We construct two point set A such that its intersection with the unit circle is a Bernstein subset of the unit circle. Let $\{L_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all straight lines in $\mathbb{R}^2$. Let $\{P_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all perfect subsets of unit circle. We will define inductively a sequence $\{A_\xi : \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}^2$ and a sequence $\{y_\xi : \xi < \mathfrak{c}\}$ of points from the unit circle such that for every $\xi < \mathfrak{c}$,

(1) $|A_\xi| < \omega$;
(2) $\bigcup_{\zeta \leq \xi} A_\zeta$ does not contain three collinear points;
(3) $\bigcup_{\zeta \leq \xi} A_\zeta$ contains precisely two points of L_ξ;
(4) $P_\xi \cap \bigcup_{\zeta \leq \xi} A_\zeta \neq \emptyset$;
(5) $y_\xi \in P_\xi$;
(6) $A_\xi \cap \{y_\zeta : \zeta \leq \xi\} = \emptyset$.

The existence of the sequence $\{A_\xi : \xi < \mathfrak{c}\}$ follows exactly in the same way as in Theorem 1. Here the key observation is that for each perfect set P_ξ of unit circle there exist $\mathfrak{c}$ many straight lines passing through P_ξ and the origin.

Putting $A = \bigcup_{\xi < \mathfrak{c}} A_\xi$ we obtain a two point s-nonmeasurable set. Clearly A is of full outer measure on the unit circle. □

5. A partial two point set that is Luzin set

The following remark holds

Remark 5. *Assume $A \subseteq \mathbb{R}^2$ is two point set. Then*

(1) *A is not Bernstein;*
(2) *A is not Luzin;*
(3) *A is not Sierpiński.*

Proof. 1) Each line L is a perfect set such that $|A \cap L| = 2$, so A cannot be Bernstein.

2) Let M be a perfect meager subset of $\mathbb{R}$. Then $M \times \mathbb{R}$ is meager and $|(M \times \mathbb{R}) \cap A| = 2|M| = \mathfrak{c}$.

3) Let N be a perfect null subset of $\mathbb{R}$. Then $N \times \mathbb{R}$ is null and $|(N \times \mathbb{R}) \cap A| = 2|N| = \mathfrak{c}$. □

Theorem 6. *(CH)*

(1) *There exists partial two point set A that is strong Luzin set.*
(2) *There exists partial two point set B that is strong Sierpiński set.*

Proof. Let us focus on the Luzin set. The case of the Sierpiński is similar.

Fix $\{B_\alpha : \alpha < \omega_1\}$ base of the ideal of meager sets and let $\{D_\alpha : \alpha < \omega_1\}$ be the enumeration of Borel nonmeager sets such that each set appears ω_1 many times. We will construct a sequence $\{x_\alpha : \alpha < \omega_1\}$ satisfying the following properties:

(1) $A_\alpha = \{x_\xi : \ \xi \leq \alpha\}$ does not contain three collinear points;
(2) $x_\alpha \in D_\alpha \setminus \bigcup_{\xi<\alpha} B_\xi$.

We will show that at any α step we can pick x_α such that (1) and (2) are fulfilled. Since A_ξ is countable so is $\bigcup_{\xi<\alpha} A_\xi$. Therefore the set $\{L : L \text{ is a line and } |L \cup \bigcup_{\xi<\alpha} A_\xi| \geq 2\} = \{L : L \text{ is a line and } |L \cup \bigcup_{\xi<\alpha} A_\xi| = 2\}$ is countable. Hence both $\{L : L \text{ is a line and } |L \cup \bigcup_{\xi<\alpha} A_\xi| = 2\}$ and $\bigcup_{\xi<\alpha} B_\xi$ are meager. Consequently one can pick a point x_α from D_α that meets neither $\{L : L \text{ is a line and } |L \cup \bigcup_{\xi<\alpha} A_\xi| = 2\}$ nor $\bigcup_{\xi<\alpha} B_\xi$. Then $A_\alpha = \{x_\xi : \xi \leq \alpha\}$ does not contain three collinear points. At the end $A = \{x_\alpha : \ \alpha < \omega_1\}$ and it is a required partial two point set that is strong Luzin. □

Let us remark that sets A and B constructed in Theorem 6 are s_0 Marczewski. Moreover A is strongly null and B is strongly meager. For the definitions of strongly meager and strongly null we refer the reader to [Car].

Theorem 6 can be strengthen. If we assume that $add(\mathbb{K}) = cof(\mathbb{K}) = \kappa$ then we can construct partial two point set A such that $|A| = \kappa$ and for every Borel set B, $|B \cap A| < \kappa$ if and only if $B \in \mathbb{K}$.

The analogous observation is true in the case of null sets.

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